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Asymptotic Satellites near the Straight-Line Equilibrium Points in the Problem of Three Bodies.

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§ 1. OSCILLATING SATELLITES.

It was shown by Lagrange in a prize memoir[*] in 1772 that if two finite spheres revolve in circles about their common centre of mass, then there are three points on the line joining their centres at which an infinitesimal body would remain if it were given an initial projection so as to be instantaneously fixed with respect to the moving bodies. These points are called the *straight-line equilibrium points* of the problem of three bodies. Starting from minus infinity the order of the equilibrium points and the finite bodies is an equilibrium point, a finite body, a second equilibrium point, the other finite body, the third equilibrium point.

If the infinitesimal body is given an initial displacement from a point of equilibrium and initial conditions are so chosen that it moves in a closed orbit relatively to the moving system, it is then called an *oscillating satellite.*

The problem of the oscillating satellite has been discussed extensively. In the papers cited below,[†] the differential equations are limited to their linear terms and the orbits restricted to the plane of motion of the finite bodies.

A rigorous demonstration for the existence of periodic orbits for the oscillating satellite and a practical method for constructing them are given by Moulton in Chapter V of his "Periodic Orbits."[‡] In this memoir the differential equations are unrestricted in the number of terms which may be taken, and the orbits are not limited to two dimensions as in the previous literature. Three classes of orbits are shown to exist and they are designated according to their periods as Class A, Class B, and Class C. The orbits of Class A and Class C are of three dimensions, while those of Class B are of two dimensions. The orbits of Class C are shown to exist under special conditions which might never be realized in the problem. Their period is a multiple of the periods of Class A and Class B when these latter periods are commensurable. Practical constructions are made for the orbits of Class A and of Class B, but, owing to the complexity of the problem, no attempt has been made to determine whether orbits of Class C exist which are distinct from those of Class A and Class B.

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[*] Lagrange, "Collected Works," Vol. VI, pp. 229–324.

[†] The following references to the literature of the oscillating satellite are taken from Chapter V of Moulton's "Periodic Orbits":

Poincaré, *Les Méthodes Nouvelles de la Mécanique Céleste,* Vol. I (1892), p. 159.

Burrau, *Astronomische Nachrichten,* Nos. 3230, 3251 (1894).

Perchot and Mascart, *Bulletin Astronomique,* Vol. XII (1895), p. 329. Moulton adds, "apparently their work is vitiated by an error in establishing the existence of the solutions, and their construction fails where they stopped."

Sir George H. Darwin, *Acta Mathematica,* Vol. XXI (1897), p. 99.

Plummer, *Monthly Notices, Royal Astronomical Society,* Vol. LXIII (1903), p. 436, and Vol. LXIV (1903), p. 98.

[‡] This memoir will be cited as the "Oscillating Satellite." Another method is given in Chapter VI of the "Periodic Orbits," but we are concerned with the former method only.

§ 2. ASYMPTOTIC SATELLITES.

The object of this paper is to obtain solutions of the differential equations of motion of the infinitesimal body which will approach the periodic solutions of Class A and Class B as the time approaches plus infinity or minus infinity. When the infinitesimal body moves in an orbit defined by such solutions, it will be called an *asymptotic satellite.*

The question of the existence of solutions which are asymptotic to the orbits of Class C is not considered in this paper, owing to the fact that it has not been determined in the "Oscillating Satellite" that orbits of Class C exist which are distinct from those of Class A and Class B.

The orbits which are asymptotic to the equilibrium points themselves have been determined by Warren.* These orbits are of two dimensions and lie in the plane of motion of the finite bodies.

The form of the asymptotic solutions is that adopted by Poincaré,† each term being of the type $e^{\lambda t}P(t)$, where λ is a constant having its real part different from zero, and $P(t)$ is a constant or periodic function of t. It has been shown by Poincaré that solutions of this type will converge for all values of t, provided that certain divisors which appear in the construction of such solutions are different from zero.‡ If these divisors vanish, terms of the form $te^{\lambda t}P(t)$ will arise. If, therefore, the construction can be made so that no terms occur in t explicitly, the divisors previously mentioned are different from zero and the solutions will converge for all values of t. Hence it is sufficient to consider only the formal construction of the asymptotic solutions, and if solutions can be constructed so as to contain no terms in t explicitly, their convergence is assured by Poincaré's theorem.

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* Warren, "A Class of Asymptotic Orbits in the Problem of Three Bodies," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXVIII, No. 3 (1916), pp. 221-248.

† Poincaré, "Mécanique Celéste," Vol. I, p. 340.

‡ Poincaré, *loc. cit.* p. 341.

§ 3. THE DIFFERENTIAL EQUATIONS OF MOTION.

Let the motion of the infinitesimal body be referred to a set of rotating rectangular coordinates $\xi\eta\zeta$ of which the origin is at the centre of mass of the finite bodies, the ξ-axis is the line joining the finite bodies, and the $\xi\eta$-plane is the plane of their motion. The ξ- and η-axes rotate about the ζ-axis in the direction of the motion of the finite bodies and with the same angular velocity. The units of length, mass, and time will be taken so that the distance between the finite bodies, the sum of their masses, and the Gaussian constant respectively shall each be unity. With the units thus chosen, the mean angular motion of the system is likewise unity. Let the masses of the finite bodies be denoted by $1-\mu$ and μ, $0<\mu\leqq\frac{1}{2}$. On denoting the coordinates of the infinitesimal body by ξ, η, ζ, and differentiation with respect to t by primes, the differential equations of motion are [*]

$$\left.\begin{aligned}
&\xi''-2\eta'=\frac{\partial U}{\partial\xi},\quad \eta''+2\xi'=\frac{\partial U}{\partial\eta},\quad \zeta''=\frac{\partial U}{\partial\zeta},\\
&2U=\xi^2+\eta^2+\frac{2(1-\mu)}{r_1}+\frac{2\mu}{r_2}\\
&\quad=(1-\mu)\left(r_1^2+\frac{2}{r_1}\right)+\mu\left(r_2^2+\frac{2}{r_2}\right)-\zeta^2-\mu(1-\mu),\\
&r_1=\sqrt{(\xi+\mu)^2+\eta^2+\zeta^2},\quad r_2=\sqrt{(\xi-1+\mu)^2+\eta^2+\zeta^2}.
\end{aligned}\right\}\qquad(1)$$

The points of equilibrium are the solutions of the equations [†]

$$\frac{\partial U}{\partial\xi}=\frac{\partial U}{\partial\eta}=\frac{\partial U}{\partial\zeta}=0.$$

There are two sets of points which satisfy these equations. One set consists of the two points which are at the vertices of the two equilateral triangles on the opposite sides of the line joining the finite bodies. The orbits which are asymptotic to these points and also to the periodic oscillations near these points are discussed in another paper.[‡] The other set of points consists of three points which lie on the straight line joining the finite bodies. Let the coordinates of these points be denoted by ξ_0, 0, 0, where the particular value of ξ_0 depends upon the equilibrium point in question.* The points themselves will be denoted by (a), (b), and (c), where (a) lies between $+\infty$ and the finite mass μ, (b) between μ and $1-\mu$, and (c) between $1-\mu$ and $-\infty$.

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[*] Moulton, "Introduction to Celestial Mechanics" (1914), p. 279.

[†] Moulton, "Introduction to Celestial Mechanics," p. 290; Charlier, "Die Mechanik des Himmels," Vol. II, pp. 102–111.

[‡] This paper is now under consideration for publication.

If the infinitesimal body is given a small displacement from an equilibrium point and a small velocity with respect to the finite masses such that

$$\left.\begin{aligned} \xi &= \xi_0+\bar{x}, & \eta &= 0+\bar{y}, & \zeta &= 0+\bar{z}, \\ \xi' &= 0+\bar{x}', & \eta' &= 0+\bar{y}', & \zeta &= 0+\bar{z}', \end{aligned}\right\} \tag{2}$$

then the differential equations (1) become†

$$\left.\begin{aligned} \bar{x}''-2\bar{y}' &= \frac{\partial U}{\partial \bar{x}} = +X(\bar{x}, \bar{y}^2, \bar{z}^2), \\ \bar{y}''+2\bar{x}' &= \frac{\partial U}{\partial \bar{y}} = \bar{y}Y(\bar{x}, \bar{y}^2, \bar{z}^2), \\ \bar{z}'' &= \frac{\partial U}{\partial \bar{z}} = \bar{z}Z(\bar{x}, \bar{y}^2, \bar{z}^2), \\ U &= \tfrac{1}{2}(1-\mu)\left(r_1^2+\frac{2}{r_1}\right)+\tfrac{1}{2}\mu\left(r_2^2+\frac{2}{r_2}\right)-\tfrac{1}{2}\bar{z}^2-\tfrac{1}{2}\mu(1-\mu), \\ r_1 &= \sqrt{(\xi_0+\bar{x}+\mu)^2+\bar{y}^2+\bar{z}^2}, \quad r_2=\sqrt{(\xi_0-1+\bar{x}+\mu)^2+\bar{y}^2+\bar{z}^2}, \end{aligned}\right\} \tag{3}$$

where X, Y, and Z are power series in $\bar{x}$, $\bar{y}^2$, $\bar{z}^2$. These series converge within certain regions about the equilibrium points. ‡

§ 4. THE PERIODIC ORBITS.

In showing the existence of periodic solutions of the differential equations of the "Oscillating Satellite" which correspond to equations (3), and later in making the construction of these solutions, the transformations

$$\bar{x}=\varepsilon x, \quad \bar{y}=\varepsilon y, \quad \bar{z}=\varepsilon z, \quad t-t_0=(1+\delta)\tau \tag{4}$$

are made, where ε is an arbitrary parameter and δ is determined as a function of ε so that the solutions in x, y, and z shall be periodic with the periodic 2π in τ. On denoting differentiation with respect to τ by a dot over the variables, the differential equations (3) become as a consequence of (4)§

$$\left.\begin{aligned} \ddot{x}-2(1+\delta)y &= (1+\delta)^2[X_1+X_2\varepsilon+\ldots\ldots+X_k\varepsilon^{k-1}+\ldots\ldots], \\ \ddot{y}+2(1+\delta)\dot{x} &= (1+\delta)^2[Y_1+Y_2\varepsilon+\ldots\ldots+Y_k\varepsilon^{k-1}+\ldots\ldots], \\ \ddot{z} &= (1+\delta)^2[Z_1+Z_2\varepsilon+\ldots\ldots+Z_k\varepsilon^{k-1}+\ldots\ldots], \end{aligned}\right\} \tag{5}$$

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* "Oscillating Satellite," equations (4).

† "Oscillating Satellite," equations (6).

‡ "Oscillating Satellite," § 77.

§ "Oscillating Satellite," equations (11).

where X_k, Y_k, and Z_k are homogeneous polynomials of degree k in x, y, and z. From (3) it is obvious that the X_k are even in y and z, Y_k odd in y and even in z, and Z_k even in y and odd in z. The explicit values of these terms up to $k=3$ are[*]

$$\begin{array}{lll}
X_1=(1+2A)x, & X_2=\frac{3}{2}B(-2x^2+y^2+z^2), & X_3=2C(2x^3-3xy^2-3xz^2), \\
Y_1=(1-A)y, & Y_2=3Bxy, & Y_3=\frac{3}{2}C(-4x^2y+y^3+yz^2), \\
Z_1=-Az, & Z_2=3Bxz, & Z_3=\frac{3}{2}C(-4x^2z+y^2z+z^3),
\end{array}$$

$$A=\frac{1-\mu}{r_1^{(0)^3}}+\frac{\mu}{r_2^{(0)^3}}, \quad r_1^{(0)}=+\sqrt{(\xi_0+\mu)^2},$$

$$B=\begin{smallmatrix}+\\+\\-\end{smallmatrix}\frac{1-\mu}{r_1^{(0)^4}}\begin{smallmatrix}+\\-\\-\end{smallmatrix}\frac{\mu}{r_2^{(0)^4}}, \quad r_2^{(0)}=+\sqrt{(\xi_0-1+\mu)^2},$$

$$C=\frac{1-\mu}{r_1^{(0)^5}}+\frac{\mu}{r_2^{(0)^5}}.$$

The upper, middle, or lower signs are to be taken in B according as the equilibrium point is (a), (b), or (c) respectively.

If periodic solutions of (5) exist, their periods are determined from the periods of the solutions of the linear terms of (5). The solutions of these linear terms are[†]

$$\left.\begin{array}{l}
x=K_1e^{i\sigma\tau}+K_2e^{-i\sigma\tau}+K_3e^{\rho\tau}+K_4e^{-\rho\tau}, \quad i=\sqrt{-1}, \\
y=in(K_1e^{i\sigma\tau}-K_2e^{-i\sigma\tau})+m(K_3e^{\rho\tau}-K_4e^{-\rho\tau}), \\
z=K_5\cos\sqrt{A}\tau+K_6\sin\sqrt{A}\tau, \quad n=\dfrac{\sigma^2+1+2A}{2\sigma}, \quad m=\dfrac{\rho^2-1-2A}{2\rho},
\end{array}\right\} \quad (6)$$

where $K_1, \ldots., K_6$ are the constants of integration, and σ^2 and ρ^2 are the negative and positive roots respectively of the quadratic in λ^2,

$$\lambda^4+(2-A)\lambda^2+(1-A)(1+2A)=0. \quad (7)$$

There are three real periods in these solutions, viz., $2\pi/\sqrt{A}$, $2\pi/\sigma$, and $P=2j\pi/\sqrt{A}=2k\pi/\sigma$, where j and k are positive integers and $\sqrt{A}$ and σ are commensurable.[‡] These are the periods of the orbits of Class A, Class B, and Class C respectively.

Orbits of Class A.

The periodic solutions of Class A are[§]

$$\left.\begin{array}{l}
\bar{x}_1=\varepsilon x_1=0\varepsilon+(a_1+b_1\cos 2\sqrt{A}\tau)\varepsilon^2+0\varepsilon^3+\ldots., \\
\bar{y}_1=\varepsilon y_1=0\varepsilon+(c_1\sin 2\sqrt{A}\tau)\varepsilon^2+0\varepsilon^3+\ldots., \\
\bar{z}_1=\varepsilon z_1=\left(\dfrac{1}{\sqrt{A}}\sin\sqrt{A}\tau\right)\varepsilon+0\varepsilon^2+d_1(3\sin\sqrt{A}\tau-\sin 3\sqrt{A}\tau)\varepsilon^3+\ldots., \\
\delta=\delta_1=0\varepsilon+\delta_1^{(2)}\varepsilon^2+\ldots.+\delta_1^{(2j)}\varepsilon^{2j},
\end{array}\right\} \quad (8)$$

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[*] "Oscillating Satellite," equations (53), (15), (35).

[†] "Oscillating Satellite," equations (16), (27), and (28).

[‡] In §83 of the "Oscillating Satellite" it is shown that there are infinitely many values of μ between 0 and $\frac{1}{2}$ such that $\sqrt{A}$ and σ are commensurable.

[§] "Oscillating Satellite," equations (71).

where
$$a_1=\frac{-3B}{4A(1+2A)},\qquad b_1=\frac{3B(1+3A)}{4A(1-7A+18A^2)},$$
$$c_1=\frac{-3B}{\sqrt{A}(1-7A+18A^2)},\qquad d_1=\frac{3}{64A^{5/2}}\left[\frac{3B^2(1+3A)}{1-7A+18A^2}-C\right],$$
$$\delta_1^{(2)}=-\frac{9}{16A^2}\left[\frac{3B^2(1-3A+14A^2)}{(1+2A)(1-7A+18A^2)}-C\right].$$

The expressions for $\bar{x}_1$ and $\bar{y}_1$ are power series in ϵ^2 with sums of cosines and sines respectively of even multiples of $\sqrt{A}\tau$ in the coefficients. The highest multiple of $\sqrt{A}\tau$ which appears in the coefficient of ϵ^k in each series is k. The solution for $\bar{z}_1$ is a power series in odd powers of ϵ having sums of sines of odd multiples of $\sqrt{A}\tau$ in the coefficients. The highest multiple of $\sqrt{A}\tau$ which occurs in the coefficient of ϵ^k is likewise k. The initial conditions are so chosen in the construction of these solutions that

$$\bar{z}(0)=0,\quad \dot{\bar{z}}(0)=\epsilon,$$

that is, the parameter ϵ is proportional to the initial projection of the infinitesimal body from the plane of motion. The actual projection is $\epsilon/(1+\delta_1)$.

Numerical examples of these orbits have been considered in the "Oscillating Satellite" and the following results have been obtained when the ratio of the finite masses is ten to one, or $1-\mu=10/11$, $\mu=1/11$.*

TABLE 1.

Coefficient.	Point (*a*).	Point (*b*).	Point (*c*).
A	2.548	6.510	1.082
σ^2	2.811	6.820	1.144
ρ^2	3.359	11.330	0.226
n	2.657	3.990	2.014
m	—0.747	— 0.397	—3.091
B	6.548	—10.961	—1.136
C	18.283	55.740	1.196
a_1	—0.316	0.090	0.249
b_1	0.151	— 0.036	—0.230
c_1	—0.112	0.018	0.226
d_1	—0.037	— 0.020	—0.002
$\delta_1^{(2)}$	0.184	0.467	0.001

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* This was the ratio used by Darwin in the memoir cited at the beginning of this article. The same ratio was adopted by Moulton in the "Oscillating Satellites" so that he might compare his results with those of Darwin.

Orbits of Class B.

The periodic solutions of Class B are *

$$\left.\begin{aligned}\bar{x}_2&=\epsilon x_2=(\cos\sigma\tau)\epsilon+(a_2+b_2\cos\sigma\tau+c_2\cos 2\sigma\tau)\epsilon^2+\ldots,\\ \bar{y}_2&=\epsilon y_2=(-n\sin\sigma\tau)\epsilon+(-nb_2\sin\sigma\tau+d_2\sin 2\sigma\tau)\epsilon^2+\ldots,\\ \delta&=\delta_2=0\epsilon+\delta_2^{(2)}\epsilon^2+\delta_2^{(3)}\epsilon^3+\ldots,\end{aligned}\right\}\qquad(9)$$

where a_2, b_2, c_2, d_2, $\delta_2^{(2)}$, ..., are known constants. The algebraic expressions for these constants may be found in equations (101), (103), and (107) of the "Oscillating Satellite." Their particular values for $1-\mu=10/11$, $\mu=1/11$ are to be found in Table 2. The first seven coefficients which appear in Table 1 have been used in determining the values in Table 2.

TABLE 2.

Coefficient.	Point (*a*).	Point (*b*).	Point (*c*).
a_2	—4.078	8.172	0.554
b_2	1.996	— 4.976	—0.038
c_2	2.082	— 3.196	—0.516
nb_2	5.305	—19.855	—0.077
d_2	1.250	— 1.478	—0.276
$\delta_2^{(2)}$	3.955	8.553	—1.407

The expressions for $\bar{x}_2$ and $\bar{y}_2$ in (9) are power series in ϵ with sums of cosines and sines respectively of $\sigma\tau$ in the coefficients. The highest multiple of $\sigma\tau$ which occurs in the coefficient of ϵ^k is k. The initial conditions are so chosen for the construction of these solutions that

$$\bar{x}_2(0)=\epsilon,\quad \bar{y}_2(0)=0.$$

In this case the parameter ϵ denotes the initial displacement along the ξ-axis of the infinitesimal body from an equilibrium point.

§ 5. The Equations of Variation.

Suppose now that the initial conditions for the periodic orbits are changed slightly. Then the infinitesimal body will deviate from its periodic orbit, and the amount and character of the deviation will depend upon the initial conditions chosen. The equations for the disturbed orbit, in terms of the variables x, y, and z, will be

$$x=x_j+p_j,\quad y=y_j+q_j,\quad z=z_j+r_j\dagger,\qquad(10)$$

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* "Oscillating Satellite," equations (97), (105).

† The r_1 and r_2 in this equation and in subsequent equations are different from r_1 and r_2 in equations (1) and (3).

where εx_j, εy_j, and εz_j represent the periodic orbits of Class A or Class B according as $j=1$ or 2 respectively. The additional terms p_j, q_j, and r_j are functions of τ which depend upon the initial disturbance from the periodic orbit. We desire to determine these functions so that they shall have the asymptotic form already referred to in § 2.

On substituting (10) in (5), the differential equations which define p_j, q_j, and r_j are found to be

$$\left.\begin{aligned} \ddot{p}_j-2(1+\delta_j)\dot{q}_j+(1+\delta_j)^2(P_{j1}p_j+P_{j2}q_j+P_{j3}r_j)&=(1+\delta_j)^2P_j,\\ \ddot{q}_j+2(1+\delta_j)\dot{p}_j+(1+\delta_j)^2(Q_{j1}p_j+Q_{j2}q_j+Q_{j3}r_j)&=(1+\delta_j)^2Q_j,\\ \ddot{r}_j+(1+\delta_j)^2(R_{j1}p_j+R_{j2}q_j+R_{j3}r_j)&=(1+\delta_j)^2R_j, \end{aligned}\right\} \tag{11}$$

where

$$\begin{aligned} P_{j1}&=-1-2A+6B\varepsilon x_j-6C\varepsilon^2(2x_j^2-y_j^2-z_j^2)+\ldots.,\\ P_{j2}=Q_{j1}&=-3B\varepsilon y_j+12C\varepsilon^2x_jy_j+\ldots.,\\ P_{j3}=R_{j1}&=-3B\varepsilon z_j+12C\varepsilon^2x_jz_j+\ldots.,\\ Q_{j2}&=-1+A-3B\varepsilon x_j+\tfrac{3}{2}C\varepsilon^2(4x_j^2-3y_j^2-z_j^2)+\ldots.,\\ Q_{j3}=R_{j2}&=-3C\varepsilon^2y_jz_j+\ldots.,\\ R_{j3}&=A-3B\varepsilon x_j+\tfrac{3}{2}C\varepsilon^2(4x_j^2-y_j^2-3z_j^2)+\ldots.,\\ P_j&=\tfrac{3}{2}B\varepsilon(-2p_j^2+q_j^2+r_j^2)+(\,)\varepsilon^2+\ldots.,\\ Q_j&=3B\varepsilon p_jq_j+(\,)\varepsilon^2+\ldots.,\\ R_j&=3B\varepsilon p_jr_j+(\,)\varepsilon^2+\ldots.. \end{aligned}$$

As we propose later to construct numerical solutions which are asymptotic to the particular periodic orbits mentioned above, we insert here the particular values of the various P_{jk}, Q_{jk}, and R_{jk} which are obtained by making use of the numerical values listed in Tables 1 and 2. The following results are found.

Orbits of Class A.

Equilibrium point (a).

$$\begin{aligned} P_{11}&=-6.096+\varepsilon^2(8.384-15.686\cos 2\sqrt{A}\tau)+(\,)\varepsilon^4+\ldots.,\\ P_{12}=Q_{21}&=\varepsilon^2(2.200\sin 2\sqrt{A}\tau)+(\,)\varepsilon^4+\ldots.,\\ P_{13}=R_{11}&=-\varepsilon(12.297\sin\sqrt{A}\tau)+(\,)\varepsilon^3+\ldots.,\\ Q_{12}&=1.548+\varepsilon^2(0.833+2.409\cos 2\sqrt{A}\tau)+(\,)\varepsilon^4+\ldots.,\\ Q_{13}=R_{12}&=(\,)\varepsilon^3+\ldots.\\ R_{13}&=2.548-\varepsilon^2(9.918-13.160\cos 2\sqrt{A}\tau)+(\,)\varepsilon^4+\ldots.. \end{aligned}$$

Equilibrium point (*b*).

$$P_{11}=-14.020+\varepsilon^2(19.777-23.328\cos 2\sqrt{A}\tau)+(\,)\varepsilon^4+\ldots,$$
$$P_{12}=Q_{21}=\varepsilon^2(0.592\sin 2\sqrt{A}\tau)+(\,)\varepsilon^4+\ldots,$$
$$P_{13}=R_{11}=\varepsilon(12.890\sin\sqrt{A}\tau)+(\,)\varepsilon^3+\ldots,$$
$$Q_{12}=5.510+\varepsilon^2(-3.464+5.240\cos 2\sqrt{A}\tau)+(\,)\varepsilon^4+\ldots,$$
$$Q_{13}=R_{12}=(\,)\varepsilon^3+\ldots,$$
$$R_{13}=6.510+\varepsilon^2(-16.312+18.108\cos 2\sqrt{A}\tau)+(\,)\varepsilon^4+\ldots.$$

Equilibrium point (*c*).

$$P_{11}=-3.164+\varepsilon^2(1.616-1.746\cos 2\sqrt{A}\tau)+(\,)\varepsilon^4+\ldots,$$
$$P_{12}=Q_{21}=\varepsilon^2(0.770\sin 2\sqrt{A}\tau)+(\,)\varepsilon^4+\ldots,$$
$$P_{13}=R_{11}=\varepsilon(3.275\sin\sqrt{A}\tau)+(\,)\varepsilon^3+\ldots,$$
$$Q_{12}=0.082+\varepsilon^2(0.020+0.044\cos 2\sqrt{A}\tau)+(\,)\varepsilon^4+\ldots,$$
$$Q_{13}=R_{12}=(\,)\varepsilon^3+\ldots,$$
$$R_{13}=1.082+\varepsilon^2(0.020+0.044\cos 2\sqrt{A}\tau)+\ldots.$$

Orbits of Class B.

Equilibrium point (*a*).

$$P_{21}=-6.10+\varepsilon(39.30\cos\sigma\tau)+\ldots,\qquad P_{22}=Q_{21}=\varepsilon(52.27\sin\sigma\tau)+\ldots,$$
$$Q_{22}=1.55-\varepsilon(19.64\cos\sigma\tau)+\ldots,\qquad P_{23}=Q_{23}=R_{21}=R_{22}=0,$$
$$R_{23}=2.55-\varepsilon(19.64\cos\sigma\tau)+\ldots.$$

Equilibrium point (*b*).

$$P_{21}=-14.02-\varepsilon(65.77\cos\sigma\tau)+\ldots,\qquad P_{22}=Q_{21}=-\varepsilon(131.20\sin\sigma\tau)+\ldots,$$
$$Q_{22}=5.51+\varepsilon(32.88\cos\sigma\tau)+\ldots,\qquad P_{23}=Q_{23}=R_{21}=R_{22}=0,$$
$$R_{23}=6.51+\varepsilon(32.88\cos\sigma\tau)+\ldots.$$

Equilibrium point (*c*).

$$P_{21}=-3.16-\varepsilon(6.82\cos\sigma\tau)+\ldots,\qquad P_{22}=Q_{21}=\varepsilon(6.86\sin\sigma\tau)+\ldots,$$
$$Q_{22}=0.08+\varepsilon(3.41\cos\sigma\tau)+\ldots,\qquad P_{23}=Q_{23}=R_{21}=R_{22}=0,$$
$$R_{23}=1.08+\varepsilon(3.41\cos\sigma\tau)+\ldots.$$

If we consider only the terms of (11) which are linear in p_j, q_j, and r_j, we obtain the *equations of variation.*[*] For the orbits of Class A they are

$$\left.\begin{aligned}\ddot{p}_1-2(1+\delta_1)\dot{q}_1+(1+\delta_1)^2(P_{11}p_1+P_{12}q_1+P_{13}r_1)&=0,\\ \ddot{q}_1+2(1+\delta_1)\dot{p}_1+(1+\delta_1)^2(Q_{11}p_1+Q_{12}q_1+Q_{13}r_1)&=0,\\ r_1+(1+\delta_1)^2(R_{11}p_1+R_{12}q_1+R_{13}r_1)&=0,\end{aligned}\right\}\qquad(12\text{A})$$

and for orbits of Class B, they are

$$\left.\begin{aligned}\ddot{p}_2-2(1+\delta_2)\dot{q}_2+(1+\delta_2)^2(P_{21}p_2+P_{22}q_2)&=0,\\ \ddot{q}_2+2(1+\delta_2)\dot{p}_2+(1+\delta_2)^2(Q_{21}p_2+Q_{22}q_2)&=0,\\ \ddot{r}_2+(1+\delta_2)^2R_{23}r_2&=0.\end{aligned}\right\}\qquad(12\text{B})$$

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[*] Poincaré, *loc. cit.*, Vol. I, Chap. 4.

The periodic solutions (8) and (9) are called the *generating solutions*[*] of (12A) and (12B) respectively.

§ 6. THE SOLUTIONS OF THE EQUATIONS OF VARIATION (12A).

The equations of variation are linear differential equations having periodic coefficients, the periods being the same as those of the corresponding generating solutions. Differential equations of this type were first treated by Hill[†] in his celebrated memoir on the lunar theory. A very extensive list of references to the literature of these differential equations was given by Baker in a recent memoir.[‡]

The method which we shall adopt in the construction of the solutions of these equations is the one developed by Moulton and Macmillan.[§] The form[||] of the solution is, in general

$$e^{\alpha\tau}\phi(\tau),$$

where $\phi(\tau)$ is a periodic function having the same period as the coefficients of the differential equations, and α is a constant called the *characteristic exponent.*

It has been shown by Poincaré[¶] that if the generating solutions contain an arbitrary constant which does not appear explicitly in the original differential equations, then a set of solutions of the equations of variation is obtained by taking the first partial derivatives of the generating solutions with respect to this constant. The characteristic exponent associated with this set of solutions is zero, and the solutions themselves are either periodic or consist of periodic functions plus τ times other periodic functions. The initial time t_0 is one such arbitrary constant which is present in all differential equations of dynamics in which the potential function *e.g.*, U in equations (1), does not contain the time explicitly. Differentiation of the generating solutions with respect to this constant yields a set of periodic solutions. Another such constant usually present is the scale factor of the generating solutions. Differentiation of the generating solutions with respect to this constant yields a set of solutions comprised of a periodic function plus $K\tau$, K a constant, times the periodic function obtained by differentiating with respect to t_0.

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[*] Poincaré, *loc. cit.* Vol. I, Chap. 4.

[†] G. W. Hill, "Collected Works," Vol. I, p. 243; *Acta Mathematica,* Vol. VIII, pp. 1–36.

[‡] H. F. Baker, "On Certain Linear Differential Equations of Astronomical Interest," *Philosophical Transactions of the Royal Society of London,* Series A (1916), Vol. 216, pp. 129–186.

[§] Moulton and Macmillan, "On the Solutions of Certain Types of Linear Differential Equations with Periodic Coefficients," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXIII (1911), No. 1, pp. 63–97.

[||] Floquet, *Annales Scientifiques de l'École Superieure,* Series II, Vol. XII (1883), p. 47.

[¶] Poincaré, *loc cit.,* Vol. I Chap. 4.

Let us consider first equations (12A), and those solutions which have characteristic exponents different from zero. These equations are simultaneous and must be considered together. To find their solutions let

$$p_1 = K_1 e^{\alpha_1 \tau} u_1, \quad q_1 = L_1 e^{\alpha_1 \tau} v_1, \quad r_1 = M_1 e^{\alpha_1 \tau} w_1, \tag{13}$$

where K_1, L_1, and M_1 are arbitrary constants. An existence proof, which is omitted here, would show that

$$\left.\begin{aligned} \alpha_1 &= \alpha_1^{(0)} + \alpha_1^{(2)} \varepsilon^2 + \dots + \alpha_1^{(2k)} \varepsilon^{2k} + \dots, \\ u_1 &= u_1^{(0)} + u_1^{(2)} \varepsilon^2 + \dots + u_1^{(2k)} \varepsilon^{2k} + \dots, \\ v_1 &= v_1^{(0)} + v_1^{(2)} \varepsilon^2 + \dots + v_1^{(2k)} \varepsilon^{2k} + \dots, \\ w_1 &= w_1^{(1)} \varepsilon + w_1^{(3)} \varepsilon^3 + \dots + w_1^{(2k+1)} \varepsilon^{2k+1} + \dots, \end{aligned}\right\} \tag{14}$$

where the $\alpha_1^{(2k)}$ are constants so determined that the various $u_1^{(2k)}$, $v_1^{(2k)}$, and $w_1^{(2k+1)}$ shall be periodic with the period $2\pi/\sqrt{A}$ in τ. On substituting (13) in (12A) and considering only the terms independent of ε, it is found that $\alpha_1^{(0)}$ must satisfy the biquadratic (7) if the arbitrary constants K_1, L_1, and M_1 are to have values other than the trivial ones $K_1 = L_1 = M_1 = 0$. It is shown in the "Oscillating Satellite," § 82, that this biquadratic admits two purely imaginary solutions, denoted by $+\sigma i$ and $-\sigma i$, and two real solutions, denoted by $+\rho$ and $-\rho$, for all values of μ such that $0 \leqq \mu \leqq \frac{1}{2}$, and for each equilibrium point (*a*), (*b*), and (*c*).

Since the functions u_1, v_1, and w_1 are multiplied by arbitrary constants, we may assume without loss of generality, that $u_1(0) = 1$. Hence

$$u_1^{(0)}(0) = 1, \quad u_1^{(2k)}(0) = 0 \qquad (k = 1, 2, \dots \infty). \tag{15}$$

Then $v_1(0)$ and $w_1(0)$ can be determined from the differential equations which define v_1 and w_1.

The various solutions of (12A) may be constructed by substituting (14) and (13) in (12A), and giving to $\alpha_1^{(0)}$ the values σi, $-\sigma i$, ρ, and $-\rho$ in turn. By equating the coefficients of the various powers of ε we obtain sets of differential equations which can be integrated step by step. The arbitrary constants of integration arising at each step can be uniquely determined from the conditions (15). The various $\alpha_1^{(2k)}$ can likewise be uniquely determined by the condition that $u_1^{(2k)}$, $v_1^{(2k)}$, and $w_1^{(2k+1)}$ shall be periodic with the period $2\pi/\sqrt{A}$ in τ. From Poincaré's extension to Cauchy's theorem,* it is known that these solutions will converge for ε sufficiently small numerically and for all values of τ such that $0 \leqq \tau \leqq T$, where T is an arbitrary period chosen in advance. Obviously, we may choose the period $2\pi/\sqrt{A}$.

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*Poincaré, *loc. cit.*, Vol. I, p. 58.

Let us denote the solutions of (12A) when $\alpha_1^{(0)}=\sigma i$ by

$$p_1=e^{i\sigma_1\tau}u_{11}, \quad q_1=e^{i\sigma_1\tau}iv_{11}, \quad r_1=e^{i\sigma_1\tau}iw_{11}. \tag{16}$$

The arbitrary constants are omitted here but they appear in equations (25). It is found that σ_1 is a power series in ε^2 with real constant coefficients. The functions u_{11} and v_{11} are likewise power series in ε^2, but their coefficients are sums of cosines and i times sines of even multiples of $\sqrt{A}\tau$. The function w_{11} is a power series in odd powers of ε, vanishing with ε, and the coefficients are sums of cosines and i times sines of odd multiples of $\sqrt{A}\tau$. In each of these three functions, the highest multiple of $\sqrt{A}\tau$ which appears in the coefficient of ε^k is k. So far as the computation has been carried out, we have

$$u_{11}=1+(\,)\varepsilon^2+\ldots., \quad v_{11}=n+(\,)\varepsilon^2+\ldots.,$$

$$w_{11}=\frac{3B}{4A-\sigma^2}\left[\frac{2}{\sigma}\cos\sqrt{A}\tau-\frac{i}{\sqrt{A}}\sin\sqrt{A}\tau\right]\varepsilon+(\,)\varepsilon^3+\ldots.,$$

$$\sigma_1=\sigma+\varepsilon^2\left[\frac{9B^2\{\sigma^2(1-13A)-(3+7A-22A^2)\}}{16A\sigma^3(4A-\sigma^2)}-\frac{\delta_1^{(2)}(1-A)(1+2A)}{\sigma^3}\right]+\ldots..$$

Since the differential equations (12A) do not contain i, they are unaltered by a change in the sign of i. If then we change the sign of i in the differential equations and in the initial conditions (15), we obtain solutions which differ from (16) only in the sign of i. Let this set of solutions be denoted by

$$p_1=e^{-i\sigma\tau}u_{12}, \quad q_1=-e^{-i\sigma\tau}iv_{12}, \quad r_1=-e^{-i\sigma\tau}iw_{12}, \tag{17}$$

where u_{12}, v_{12}, and w_{12} differ from u_{11}, v_{11}, and w_{11} respectively only in the sign of i. The same result would be obtained by putting $\alpha_1^{(0)}=-i\sigma$ and proceeding as in the construction of the solution (16).

Likewise by putting $\alpha_1^{(0)}$ equal, first to ρ, and then to $-\rho$, we obtain the respective solutions

$$p_1=e^{\rho_1\tau}u_{13}, \qquad q_1=e^{\rho_1\tau}v_{13}, \qquad r_1=e^{\rho_1\tau}w_{13}, \tag{18}$$

and

$$p_1=e^{-\rho_1\tau}u_{14}, \quad q_1=-e^{-\rho_1\tau}v_{14}, \quad r_1=-e^{-\rho_1\tau}w_{14}, \tag{19}$$

where ρ_1 is a power series in ε^2 with real constant coefficients. Further u_{13}, u_{14}; v_{13}, v_{14}; w_{13}, w_{14} are similar to u_{11}, v_{11}, and w_{11} respectively, except that the coefficients of the former functions are all real; also

$$u_{14}(\tau)=u_{13}(-\tau), \quad v_{14}(\tau)=v_{13}(-\tau), \quad w_{14}(\tau)=w_{13}(-\tau).$$

The computation for the first terms of these series gives

$$u_{13}=1+(\,)\varepsilon^2+\ldots., \quad v_{13}=m+(\,)\varepsilon^2+\ldots.,$$

$$w_{13}=-\frac{3B}{4A+\rho^2}\left[\frac{2}{\rho}\cos\sqrt{A}\tau-\frac{1}{\sqrt{A}}\sin\sqrt{A}\tau\right]\varepsilon+(\,)\varepsilon^3+\ldots.,$$

$$\rho_1=\rho+\varepsilon^2\left[\frac{9B^2\{\rho^2(1-13A)+3+7A-22A^2\}}{14A\rho^3(4A+\rho^2)}+\frac{\delta_1^{(2)}(1-A)(1+2A)}{\rho^3}\right]+\ldots..$$

Obviously, the solutions (19) differ from (18) only in the sign of τ, q_1, and r_1. This property can be derived directly from the differential equations, which we proceed to show. We shall show that if

$$p_j=F_{j1}(\tau), \quad q_j=F_{j2}(\tau), \quad r_j=F_{j3}(\tau) \tag{20}$$

are solutions of the differential equations (11), then

$$p_j=F_{j1}(-\tau), \quad q_j=-F_{j2}(-\tau), \quad r_j=-F_{j3}(-\tau)$$

are likewise solutions. If this property holds for equations (11) it will evidently hold for equations (12A) or (12B), *i. e.*, when the right members of (11) are omitted.

Let us consider the differential equations (1). Since U is even in η and ζ, it follows that $\frac{\partial U}{\partial \xi}$ is even in η and ζ, $\frac{\partial U}{\partial \eta}$ is odd in η and even in ζ, and $\frac{\partial U}{\partial \zeta}$ is even in η and odd in ζ. After the substitutions (2) are made, the right members of (3) are functions of $\bar{y}$ and $\bar{z}$ as indicated. As the substitutions (4) do not alter the parity of y and z, then it follows that the right members of (5) have the same parity in y and z that the corresponding right members have in $\bar{y}$ and $\bar{z}$. When the substitutions (10) are made in (5) we obtain (11), and the terms of (11) which carry $(1+\delta_j)^2$ as a factor in each equation arise from the corresponding right members of (5). Let these terms be denoted by (11, 1), (11, 2), and (11, 3) respectively. Then they have the following properties:

(11, 1) is even in $\{y_j, q_j\}$, and even in $\{z_j, r_j\}$,
(11, 2) is odd in $\{y_j, q_j\}$, and even in $\{z_j, r_j\}$,
(11, 3) is even in $\{y_j, q_j\}$, and odd in $\{z_j, r_j\}$,

where the braces $\{\}$ denote that the variables within are to be considered together. Further, since $\bar{y}$ is a factor of the right member of the second equation in (3), it follows that (11, 2) carries a factor which is odd in $\{y_j, q_j\}$. Similarly, since $\bar{z}$ is a factor of the right members of the third equation in (3), then (11, 3) carries a factor which is odd in $\{z_j, q_j\}$.

Now, if the signs of y_j, z_j, q_j, and r_j be changed in the above expressions, then (11, 1) remains unchanged while (11, 2) and (11, 3) change signs. On examining the generating solutions (8) and (9) we observe that changing the signs of y_j and z_j is equivalent to changing the sign of τ in these solutions. Thus, if we change the signs of τ, q_j, and r_j, (11, 1) remains unchanged while (11, 2) and (11, 3) change signs. If these changes are made in all the terms of (11), the first equation remains unchanged while each term of the remaining equations changes sign and the factor -1 can be divided out. Thus the equations (11) remain unchanged if we change the signs of τ, q_j and r_j.

Suppose now that (20) is a set of solutions of (11). Let

$$\bar{p}_j=p_j, \quad \bar{q}_j=-q_j, \quad \bar{r}_j=-r_j, \quad \bar{\tau}=-\tau. \tag{21}$$

Then the differential equations obtained by making these substitutions in (11) are identically the same in $\bar{p}_j$, $\bar{q}_j$, $\bar{r}_j$, and $\bar{\tau}$ as (11) are in the former variables. With the same initial conditions for $\bar{p}_j$, $\bar{q}_j$, $\bar{r}_j$, and $\bar{\tau}$ as for equations (11), we obtain the solutions

$$\bar{p}_j=F_{j1}(\bar{\tau}), \quad \bar{q}_j=F_{j2}(\bar{\tau}), \quad \bar{r}_j=F_{j3}(\bar{\tau}),$$

where F_{j1}, F_{j2}, and F_{j3} are the same functions as in (20). On restoring the former variables by (21) we have

$$p_j=F_{j1}(-\tau), \quad q_j=-F_{j2}(-\tau), \quad r_j=-F_{j3}(-\tau)$$

as solutions of (11). Therefore if a set of solutions of (11) is known, another set can be obtained by changing the signs of τ, q_j, and r_j in the former set.

Let us next consider the two remaining sets of solutions of (12A). These solutions have characteristic exponents zero and may be obtained, as already referred to in this section, by differentiating the generating solutions (8) with respect to the arbitrary constants t_0 and ε. Taking the constant t_0 first, we have the solutions

$$p_1=\frac{\partial\bar{x}_1}{\partial t_0}=\frac{\partial\bar{x}_1}{\partial\tau}\,\frac{\partial\tau}{\partial t_0}=-\frac{1}{1+\delta_1}\Big[(-2b_1\sqrt{A}\sin 2\sqrt{A}\tau)\varepsilon^2+(\,)\varepsilon^4+\dots\Big]$$
$$=-\frac{1}{1+\delta_1}u_{15},$$

$$q_1=\frac{\partial\bar{y}_1}{\partial t_0}=\frac{\partial\bar{y}_1}{\partial\tau}\,\frac{\partial\tau}{\partial t_0}=-\frac{1}{1+\delta_1}\Big[(2c_1\sqrt{A}\cos 2\sqrt{A}\tau)\varepsilon^2+(\,)\varepsilon^4+\dots\Big]$$
$$=-\frac{1}{1+\delta_1}v_{15},$$

$$r_1=\frac{\partial\bar{z}_1}{\partial t_0}=\frac{\partial\bar{z}_1}{\partial\tau}\,\frac{\partial\tau}{\partial t_0}=-\frac{1}{1+\delta_1}\Big[(\cos\sqrt{A}\tau)\varepsilon+(\,)\varepsilon^3+\dots\Big]=-\frac{1}{1+\delta_1}w_{15}.$$

As these solutions are later multiplied by arbitrary constants, we may neglect the factor $-\dfrac{1}{1+\delta_1}$ and consider

$$p_1=u_{15}, \quad q_1=v_{15}, \quad r_1=w_{15} \tag{22}$$

as the solutions. Similarly, on differentiating (8) with respect to ε we obtain

$$p_1=\frac{\partial\bar{x}_1}{\partial\varepsilon}, \quad q_1=\frac{\partial\bar{y}_1}{\partial\varepsilon}, \quad r_1=\frac{\partial\bar{z}_1}{\partial\varepsilon}.$$

Since $\tau=(t-t_0)/(1+\delta_1)$ and δ_1 is a function of ε, then the constant ε enters the generating solutions not only explicitly but implicitly through δ_1 and τ. Hence

$$\frac{\partial \bar{x}_1}{\partial \varepsilon} = \left(\frac{\partial \bar{x}_1}{\partial \varepsilon}\right) + \frac{\partial \bar{x}_1}{\partial \tau}\frac{\partial \tau}{\partial \delta_1}\frac{\partial \delta_1}{\partial \varepsilon}, \quad \frac{\partial \bar{y}_1}{\partial \varepsilon} = \left(\frac{\partial \bar{y}_1}{\partial \varepsilon}\right) + \frac{\partial \bar{y}_1}{\partial \tau}\frac{\partial \tau}{\partial \delta_1}\frac{\partial \delta_1}{\partial \varepsilon},$$

$$\frac{\partial \bar{z}_1}{\partial \varepsilon} = \left(\frac{\partial \bar{z}_1}{\partial \varepsilon}\right) + \frac{\partial \bar{z}_1}{\partial \tau}\frac{\partial \tau}{\partial \delta_1}\frac{\partial \delta_1}{\partial \varepsilon},$$

where the parentheses around the partial derivatives denote that the differentiation is performed only in so far as ε occurs explicitly. When the differentiations are performed, we have

$$p_1 = u_{16} + K\tau u_{15}, \quad q_1 = v_{16} + K\tau v_{15}, \quad r_1 = w_{16} + K\tau w_{15}, \tag{23}$$

where

$$\begin{aligned} u_{16} &= \left(\frac{\partial \bar{x}_1}{\partial \varepsilon}\right) = 2(a_1 + b_1 \cos 2\sqrt{A}\tau)\varepsilon + ()\varepsilon^3 + \ldots., \\ v_{16} &= \left(\frac{\partial \bar{y}_1}{\partial \varepsilon}\right) = 2(c_1 \sin 2\sqrt{A}\tau)\varepsilon + ()\varepsilon^3 + \ldots., \\ w_{16} &= \left(\frac{\partial \bar{z}_1}{\partial \varepsilon}\right) = \frac{1}{\sqrt{A}} \sin \sqrt{A}\tau + ()\varepsilon^2 + \ldots., \\ K &= -\frac{1}{1+\delta_1}[2\delta_1^{(2)}\varepsilon + ()\varepsilon^3 + \ldots.]. \end{aligned}$$

It remains now to show that the solutions which we have obtained, viz., (16), (17), (18), (19), (22), and (23) constitute a fundamental set. The criterion for a fundamental set is that the determinant formed from these solutions and their first derivatives with respect to τ shall be different from zero. This determinant is

$$\Delta_1 = \begin{vmatrix} e^{i\sigma_1\tau}u_{11}, & e^{-i\sigma_1\tau}u_{12}, & e^{\rho_1\tau}u_{13}, & e^{-\rho_1\tau}u_{14}, & u_{15}, & u_{16} + K\tau u_{15}, \\ e^{i\sigma_1\tau}(i\sigma_1 u_{11} + \dot{u}_{11}), & e^{-i\sigma_1\tau}(-i\sigma_1 u_{12} + \dot{u}_{12}), & e^{\rho_1\tau}(\rho_1 u_{13} + \dot{u}_{13}), & e^{-\rho_1\tau}(-\rho_1 u_{14} + \dot{u}_{14}), & \dot{u}_{15}, & \dot{u}_{16} + K(\tau\dot{u}_{15} + u_{15}), \\ e^{i\sigma_1\tau}iv_{11}, & e^{-i\sigma_1\tau}(-iv_{12}), & e^{\rho_1\tau}v_{13}, & e^{-\rho_1\tau}(-v_{14}), & v_{15}, & v_{16} + K\tau v_{15}, \\ e^{i\sigma_1\tau}(-\sigma_1 v_{11} + i\dot{v}_{11}), & e^{-i\sigma_1\tau}(\sigma_1 v_{12} - i\dot{v}_{12}), & e^{\rho_1\tau}(\rho_1 v_{13} + \dot{v}_{13}), & e^{-\rho_1\tau}(\rho_1 v_{14} - \dot{v}_{14}), & \dot{v}_{15}, & \dot{v}_{16} + K(\tau\dot{v}_{15} + v_{15}), \\ e^{i\sigma_1\tau}iw_{11}, & e^{-i\sigma_1\tau}(-iw_{12}), & e^{\rho_1\tau}w_{13}, & e^{-\rho_1\tau}(-w_{14}), & w_{15}, & w_{16} + K\tau w_{15}, \\ e^{i\sigma_1\tau}(-\sigma_1 w_{11} + i\dot{w}_{11}), & e^{-i\sigma_1\tau}(\sigma_1 w_{12} - i\dot{w}_{12}), & e^{\rho_1\tau}(\rho_1 w_{13} + \dot{w}_{13}), & e^{-\rho_1\tau}(\rho_1 w_{14} - \dot{w}_{14}), & \dot{w}_{15}, & \dot{w}_{16} + K(\tau\dot{w}_{15} + w_{15}). \end{vmatrix} \tag{24}$$

It is a constant,[*] and its value can be determined with the least difficulty by putting $\tau = 0$. Thus we obtain

$$\Delta_1 = 4i\varepsilon(m\rho + n\sigma)(m\sigma - n\rho) + \text{ terms of higher degree in } \varepsilon.$$

It is shown in the "Oscillating Satellite," equations (36), that $m\rho + n\sigma$ and $m\sigma - n\rho$ are different from zero. Hence the determinant Δ_1 is different from zero for ε not zero, but sufficiently small numerically. Therefore the solutions which have been obtained for (12A) constitute a fundamental set, and the most general solutions are

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[*] Moulton, "Periodic Orbits," Chap. 1, Sec. 18.

$$\left.\begin{aligned}
p_1 &= K_1 e^{i\sigma_1\tau} u_{11} + K_2 e^{-i\sigma_1\tau} u_{12} + K_3 e^{\rho_1\tau} u_{13} + K_4 e^{-\rho_1\tau} u_{14} + K_5 u_{15} \\
&\qquad + K_6 (u_{16} + K\tau u_{15}), \\
q_1 &= K_1 e^{i\sigma_1\tau} i v_{11} - K_2 e^{-i\sigma_1\tau} i v_{12} + K_3 e^{\rho_1\tau} v_{13} - K_4 e^{-\rho_1\tau} v_{14} + K_5 v_{15} \\
&\qquad + K_6 (v_{16} + K\tau v_{15}), \\
r_1 &= K_1 e^{i\sigma_1\tau} i w_{11} - K_2 e^{-i\sigma_1\tau} i w_{12} + K_3 e^{\rho_1\tau} w_{13} - K_4 e^{-\rho_1\tau} w_{14} + K_5 w_{15} \\
&\qquad + K_6 (w_{16} + K\tau w_{15}),
\end{aligned}\right\} \quad (25)$$

where $K_1, \ldots, K_6$ are arbitrary constants.

§ 7. THE SOLUTIONS OF THE EQUATIONS OF VARIATION (12B).

The solutions of the equations (12B) are obtained in the same way as the preceding solutions were found. The construction is simplified by the fact that the last equation is independent of the first two.

The two sets of solutions of the first two equations of (12B) which have characteristic exponents different from zero, are

$$p_2 = e^{\rho_2\tau} u_{21}, \quad q_2 = e^{\rho_2\tau} v_{21}, \qquad (26)$$

and

$$p_2 = e^{-\rho_2\tau} u_{22}, \quad q_2 = -e^{-\rho_2\tau} v_{22}, \qquad (27)$$

where ρ_2 is a power series in ε with real coefficients, the terms of lower degrees being

$$\rho_2 = \rho + 0\varepsilon + \rho_2^{(2)} \varepsilon^2 + \rho_2^{(3)} \varepsilon^3 + \ldots\ldots$$

The functions u_{21}, v_{21}, u_{22}, and v_{22} are power series in ε with sums of sines and cosines of multiples of $\sigma\tau$ in the coefficients, the highest multiple of $\sigma\tau$ which occurs in the coefficient of ε^k being k. All the numerical coefficients in these series are real. From the property of the differential equations (11) which was proved in the preceding section, it follows that the solutions (27) are obtained from (26) by changing the signs of τ and q_2. Thus u_{22} and v_{22} differ from u_{21} and v_{21} respectively only in the signs of the sines.

Since these solutions (26) and (27) are multiplied later by arbitrary constants, we may take $u_{21}(0) = 1$. The algebraic forms of the functions u_{21} and v_{21} are

$$\begin{gathered}
u_{21} = 1 + (a_{21} \cos \sigma\tau + b_{21} \sin \sigma\tau)\varepsilon + (\,)\varepsilon^2 + \ldots, \\
v_{21} = m + (a_{22} \cos \sigma\tau + b_{22} \sin \sigma\tau)\varepsilon + (\,)\varepsilon^2 + \ldots, \\
u_{22}(\tau) = u_{21}(-\tau), \quad v_{22}(\tau) = v_{21}(-\tau).
\end{gathered}$$

The two remaining solutions of the first two equations of (12B) are obtained by differentiating the generating solutions (9) with respect to t_0 and ε. Differentiation with respect to t_0 gives

$$p_2 = \frac{\partial \bar{x}_2}{\partial t_0} = -\frac{1}{1+\delta_2}\,\frac{\partial \bar{x}_2}{\partial \tau}, \quad q_2 = \frac{\partial \bar{y}_2}{\partial t_0} = -\frac{1}{1+\delta_2}\,\frac{\partial \bar{y}_2}{\partial \tau},$$

and, since the multiplier $-\dfrac{1}{1+\delta_2}$ may be dropped, the solutions become

$$\left.\begin{aligned} p_2 = u_{23} &= \frac{\partial \bar{x}_2}{\partial \tau} = (-\sigma \sin \sigma\tau)\varepsilon - (b_2\sigma \sin \sigma\tau + 2c_2\sigma \sin 2\sigma\tau)\varepsilon^2 + \ldots., \\ q_2 = v_{23} &= \frac{\partial \bar{y}_2}{\partial \tau} = (-n\sigma \cos \sigma\tau)\varepsilon - (nb_2\sigma \cos \sigma\tau - 2d_2\sigma \cos 2\sigma\tau)\varepsilon^2 + \ldots.. \end{aligned}\right\} \quad (28)$$

Differentiation with respect to ε gives

$$p_2 = u_{24} + L\tau u_{23}, \quad q_2 = v_{24} + L\tau v_{23}, \qquad (29)$$

where

$$\begin{aligned} u_{24} &= \left(\frac{\partial \bar{x}_2}{\partial \varepsilon}\right) = \cos \sigma\tau + 2(a_2 + b_2 \cos \sigma\tau + c_2 \cos 2\sigma\tau)\varepsilon + \ldots., \\ v_{24} &= \left(\frac{\partial \bar{y}_2}{\partial \varepsilon}\right) = -n \sin \sigma\tau - 2(nb_2 \sin \sigma\tau - d_2 \sin 2\sigma\tau)\varepsilon + \ldots., \\ L &= -\frac{1}{1+\delta_2}\,(2\delta_2^{(2)}\varepsilon + 3\delta_2^{(3)}\varepsilon^2 + \ldots.). \end{aligned}$$

The determinant of these solutions (26), (27), (28), and (29) and their first derivatives with respect to τ is

$$\Delta_2 = \begin{vmatrix} e^{\rho_2\tau}u_{21}, & e^{-\rho_2\tau}u_{22}, & u_{23}, & u_{24} + L\tau u_{23}, \\ e^{\rho_2\tau}(\rho_2 u_{21} + \dot{u}_{21}), & e^{-\rho_2\tau}(-\rho_2 u_{22} + \dot{u}_{22}), & \dot{u}_{23}, & \dot{u}_{24} + L(\tau\dot{u}_{23} + u_{23}), \\ e^{\rho_2\tau}v_{21}, & e^{-\rho_2\tau}(-v_{22}), & v_{23}, & v_{24} + L\tau v_{23}, \\ e^{\rho_2\tau}(\rho_2 v_{21} + \dot{v}_{21}), & e^{-\rho_2\tau}(\rho_2 v_{22} - \dot{v}_{22}), & \dot{v}_{23}, & \dot{v}_{24} + L(\tau\dot{v}_{23} + v_{23}), \end{vmatrix} \qquad (30)$$

$$= 2\sigma\varepsilon(m\rho + n\sigma)(m\sigma - n\rho) + \text{terms of higher degree in } \varepsilon.$$

As in the former determinant Δ_1, equation (24), the factors $m\rho + n\sigma$ and $m\sigma - n\rho$ are different from zero, and since $\sigma \neq 0$ it follows that $\Delta_2 \neq 0$ for ε not zero, but sufficiently small numerically. Hence the solutions (26), (27), (28), and (29) constitute a fundamental set of solutions of the first two equations, and the most general solutions are

$$\left.\begin{aligned} p_2 &= L_1 e^{\rho_2\tau}u_{21} + L_2 e^{-\rho_2\tau}u_{22} + L_3 u_{23} + L_4(u_{24} + L\tau u_{23}), \\ q_2 &= L_1 e^{\rho_2\tau}v_{21} - L_2 e^{-\rho_2\tau}v_{22} + L_3 v_{23} + L_4(v_{24} + L\tau v_{23}), \end{aligned}\right\} \quad (31)$$

where $L_1, \ldots., L_4$ are arbitrary constants.

The general solution of the last equation of (12B) is readily obtained, and it has the form

$$r_2 = L_5 e^{i\omega\tau} w_{21} + L_6 e^{-i\omega\tau} w_{22}, \tag{32}$$

where L_5 and L_6 are arbitrary constants, and ω, w_{21}, and w_{22} are power series in ε of the form

$$\begin{aligned} w_{21} &= 1 + \frac{3B\varepsilon}{\sigma^2(\sigma^2-4A)}\left(\sigma^2 \cos \sigma\tau - 2i\sqrt{A} \sin \sigma\tau - \sigma^2\right) + \ldots, \\ w_{22} &= 1 + \frac{3B\varepsilon}{\sigma^2(\sigma^2-4A)}\left(\sigma^2 \cos \sigma\tau + 2i\sqrt{A} \sin \sigma\tau - \sigma^2\right) + \ldots, \\ \omega &= \sqrt{A} + 0\varepsilon + \omega_2\varepsilon^2 + \ldots . \end{aligned}$$

Since L_5 and L_6 are arbitrary, the initial values of w_{21} and w_{22} may be chosen so that $w_{21}(0) = w_{22}(0) = 1$. The coefficients of the various powers of ε in these functions are sums of cosines and i times sines of multiples of $\sigma\tau$, the highest multiple which occurs in the coefficient of ε^k being k. Further, w_{21} and w_{22} differ only in the sign of i. In the series for ω, all the coefficients are real. The determinant of the two solutions in (32) and their derivatives with respect to τ is

$$\begin{aligned} \Delta_8 &= \begin{vmatrix} e^{i\omega\tau} w_{21}, & e^{-i\omega\tau} w_{22}, \\ e^{i\omega\tau}(i\omega w_{21} + \dot{w}_{21}), & e^{-i\omega\tau}(-i\omega w_{21} + \dot{w}_{21}), \end{vmatrix} \\ &= -2i\omega + \text{terms in } \varepsilon, \end{aligned}$$

and this is different from zero for $|\varepsilon|$ sufficiently small. Hence the solutions (32) constitute a fundamental set of solutions of the last equation of (12B).

§ 8. THE CONSTRUCTION OF ASYMPTOTIC SOLUTIONS.

Having determined the solutions of the equations of variation, we shall now proceed to construct the asymptotic solutions of (11). We shall consider first the solutions which approach zero as τ approaches $+\infty$, and then show how to obtain from these the solutions which approach zero as τ approaches $-\infty$.

In making the constructions, it is convenient to introduce a parameter γ by the substitutions

$$p_j = \bar{p}_j\gamma, \quad q_j = \bar{q}_j\gamma, \quad r_j = \bar{r}_j\gamma, \tag{33}$$

where $\bar{p}_j$, $\bar{q}_j$, and $\bar{r}_j$ are new dependent variables. Then, since the asymptotic solutions are to be constructed as power series in γ, we put

$$\left.\begin{aligned} \bar{p}_j = p_j^{(0)} + p_j^{(1)}\gamma + p_j^{(2)}\gamma^2 + \ldots, \quad \bar{q}_j = q_j^{(0)} + q_j^{(1)}\gamma + q_j^{(2)}\gamma^2 + \ldots, \\ \bar{r}_j = r_j^{(0)} + r_j^{(1)}\gamma + r_j^{(2)}\gamma^2 + \ldots . \end{aligned}\right\} \tag{34}$$

When (34) and (33) are substituted in (11), the factor γ will divide out. Let the resulting differential equations be denoted by (11′). On equating the coefficients of the various powers of γ in these equations, we obtain sets of differential equations which define the functions $p_j^{(k)}$, $q_j^{(k)}$, and $r_j^{(k)}$ in (34). In order that the solutions shall be asymptotic we must impose the condition

(C_1), that each term of the solutions for the various $p_j^{(k)}$, $q_j^{(k)}$, and $r_j^{(k)}$ shall contain $e^{-\rho_j\tau}$ as a factor.

As an arbitrary constant arises at each step of the integration we may impose the further condition

(C_2), that $p_j(0)=\alpha_j$,

from which it follows that

$$p_j^{(0)}(0)=\alpha_j,\quad p_j^{(k)}(0)=0,\quad (k=1,\ldots.\infty).$$

Asymptotic Orbits of Class A.

Let us consider first the orbits which are asymptotic to the periodic orbits of Class A. We put $j=1$ and consider the various differential equations obtained by equating the coefficients of the same powers of γ in (11′).

For the terms in $p_1^{(0)}$, $q_1^{(0)}$, and $r_1^{(0)}$ we obtain a set of differential equations which are the same as (12A), except for the superscript 0. The general solutions are the same as (25), and when the condition (C_1) is imposed all the constants of integration must be zero except the one associated with $e^{-\rho_1\tau}$. From condition (C_2) it follows that this constant must have the value α_1. Therefore the desired solutions are

$$p_1^{(0)}=\alpha_1 e^{-\rho_1\tau}u_{14},\quad q_1^{(0)}=-\alpha_1 e^{-\rho_1\tau}v_{14},\quad r_1^{(0)}=-\alpha_1 e^{-\rho_1\tau}w_{14}. \tag{35}$$

The differential equations which are obtained from equating the coefficients of γ to the first degree in (11′) have the same left members as (12A), except for the superscript 1 on the variables. On denoting the corresponding right members by $P_1^{(1)}$, $Q_1^{(1)}$, and $R_1^{(1)}$, we find that

$$P_1^{(1)}=\varepsilon\alpha_1^2 e^{-2\rho_1\tau}U_{12}^{(1)},\quad Q_1^{(1)}=\varepsilon\alpha_1^2 e^{-2\rho_1\tau}V_{12}^{(1)},\quad R_1^{(1)}=\varepsilon\alpha_1^2 e^{-2\rho_1\tau}W_{12}^{(1)},$$

where $U_{12}^{(1)}$, $V_{12}^{(1)}$, and $W_{12}^{(1)}$ are power series of the same form as u_{14}, v_{14}, and w_{14} respectively. The complementary functions of these differential equations are the same as (25), but let us suppose that the arbitrary multipliers are

$k_1^{(1)}, \ldots, k_6^{(1)}$. The particular integrals can be obtained by varying these multipliers. Thus

$$\left.\begin{aligned}
&\dot{k}_1^{(1)}e^{i\sigma_1\tau}u_{11}+\dot{k}_2^{(1)}e^{-i\sigma_1\tau}u_{12}+\dot{k}_3^{(1)}e^{\rho_1\tau}u_{13}+\dot{k}_4^{(1)}e^{-\rho_1\tau}u_{14}+\dot{k}_5^{(1)}u_{15}\\
&\qquad\qquad\qquad\qquad\qquad\qquad+\dot{k}_6^{(1)}(u_{16}+K\tau u_{15})=0,\\
&\dot{k}_1^{(1)}e^{i\sigma_1\tau}(i\sigma_1u_{11}+\dot{u}_{11})-\dot{k}_2^{(1)}e^{-i\sigma_1\tau}(i\sigma_1u_{12}-\dot{u}_{12})+\dot{k}_3^{(1)}e^{\rho_1\tau}(\rho_1u_{13}+\dot{u}_{13})\\
&\qquad-\dot{k}_4^{(1)}e^{-\rho_1\tau}(\rho_1u_{14}-\dot{u}_{14})+\dot{k}_5^{(1)}\dot{u}_{15}+\dot{k}_6^{(1)}[\dot{u}_{16}+K(u_{15}+\tau\dot{u}_{15})]=P_1^{(1)},\\
&\dot{k}_1^{(1)}e^{i\sigma_1\tau}iv_{11}-\dot{k}_2^{(1)}e^{-i\sigma_1\tau}iv_{12}+\dot{k}_3^{(1)}e^{\rho_1\tau}v_{13}-\dot{k}_4^{(1)}e^{-\rho_1\tau}v_{14}+\dot{k}_5^{(1)}v_{15}\\
&\qquad\qquad\qquad\qquad\qquad\qquad+\dot{k}_6^{(1)}(v_{16}+K\tau v_{15})=0,\\
&\dot{k}_1^{(1)}e^{i\sigma_1\tau}(-\sigma_1v_{11}+i\dot{v}_{11})+\dot{k}_2^{(1)}e^{-i\sigma_1\tau}(\sigma_1v_{12}-i\dot{v}_{12})+\dot{k}_3^{(1)}e^{\rho_1\tau}(\rho_1v_{13}+\dot{v}_{13})\\
&\qquad+\dot{k}_4^{(1)}e^{-\rho_1\tau}(\rho_1v_{14}-\dot{v}_{14})+\dot{k}_5^{(1)}\dot{v}_{15}+\dot{k}_6^{(1)}[\dot{v}_{16}+K(v_{15}+\tau\dot{v}_{15})]=Q_1^{(1)},\\
&\dot{k}_1^{(1)}e^{i\sigma_1\tau}iw_{11}-\dot{k}_2^{(1)}e^{-i\sigma_1\tau}iw_{12}+\dot{k}_3^{(1)}e^{\rho_1\tau}w_{13}-\dot{k}_4^{(1)}e^{-\rho_1\tau}w_{14}+\dot{k}_5^{(1)}w_{15}\\
&\qquad\qquad\qquad\qquad\qquad\qquad+\dot{k}_6^{(1)}(w_{16}+K\tau w_{15})=0,\\
&\dot{k}_1^{(1)}e^{i\sigma_1\tau}(-\sigma_1w_{11}+i\dot{w}_{11})+\dot{k}_2^{(1)}e^{-i\sigma_1\tau}(\sigma_1w_{12}-i\dot{w}_{12})+\dot{k}_3^{(1)}e^{\rho_1\tau}(\rho_1w_{13}+\dot{w}_{13})\\
&\qquad+\dot{k}_4^{(1)}e^{-\rho_1\tau}(\rho_1w_{14}-\dot{w}_{14})+\dot{k}_5^{(1)}\dot{w}_{15}+\dot{k}_6^{(1)}[\dot{w}_{16}+K(w_{15}+\tau\dot{w}_{15})]=R_1^{(1)}.
\end{aligned}\right\}\quad(36)$$

The determinant of the coefficients of $\dot{k}_1^{(1)}, \ldots, \dot{k}_6^{(1)}$ is Δ_1, equation (24), and since it is different from zero for ε not zero but sufficiently small numerically these equations can be solved for $\dot{k}_1^{(1)}, \ldots, \dot{k}_6^{(1)}$. Thus

$$\dot{k}_l^{(1)}=\frac{\Delta_{1l}^{(1)}}{\Delta_1},\qquad (l=1,\ldots,6),\tag{37}$$

where $\Delta_{1l}^{(1)}$ is the determinant formed by replacing the elements of the l-th column of Δ_1, with 0, $P_1^{(1)}$, 0, $Q_1^{(1)}$, 0, and $R_1^{(1)}$ respectively. Since $P_1^{(1)}$, $Q_1^{(1)}$, and $R_1^{(1)}$ contain no terms in $e^{\pm i\sigma_1\tau}$ or $e^{\pm\rho_1\tau}$, the integrations of (37) for $k_1^{(1)}, \ldots, k_4^{(1)}$ will yield no terms in τ explicitly. Such terms will occur, however, in the integrations for $k_5^{(1)}$ and $k_6^{(1)}$, but when they are substituted in the complementary functions they will cancel off. The complete solutions are thus found to be

$$\left.\begin{aligned}
p_1^{(1)}&=K_1^{(1)}e^{i\sigma_1\tau}u_{11}+K_2^{(1)}e^{-i\sigma_1\tau}u_{12}+K_8^{(1)}e^{\rho_1\tau}u_{13}+K_4^{(1)}e^{-\rho_1\tau}u_{14}+K_5^{(1)}u_{15}\\
&\qquad\qquad\qquad\qquad+K_6^{(1)}(u_{16}+K\tau u_{15})+\varepsilon\alpha_1^2e^{-2\rho_1\tau}u_{12}^{(1)},\\
q_1^{(1)}&=K_1^{(1)}e^{i\sigma_1\tau}iv_{11}-K_2^{(1)}e^{-i\sigma_1\tau}iv_{12}+K_3^{(1)}e^{\rho_1\tau}v_{13}-K_4^{(1)}e^{-\rho_1\tau}v_{14}+K_5^{(1)}v_{15}\\
&\qquad\qquad\qquad\qquad+K_6^{(1)}(v_{16}+K\tau v_{15})+\varepsilon\alpha_1^2e^{-2\rho_1\tau}v_{12}^{(1)},\\
r_1^{(1)}&=K_1^{(1)}e^{i\sigma_1\tau}iw_{11}-K_2^{(1)}e^{-i\sigma_1\tau}iw_{12}+K_3^{(1)}e^{\rho_1\tau}w_{13}-K_4^{(1)}e^{-\rho_1\tau}w_{14}+K_5^{(1)}w_{15}\\
&\qquad\qquad\qquad\qquad+K_6^{(1)}(w_{16}+K\tau w_{15})+\varepsilon\alpha_1^2e^{-2\rho_1\tau}w_{12}^{(1)},
\end{aligned}\right\}\quad(38)$$

where $K_1^{(1)}, \ldots, K_6^{(1)}$ are the constants of integration, and $u_{12}^{(1)}$, $v_{12}^{(1)}$, and $w_{12}^{(1)}$ are functions similar to u_{14}, v_{14}, and w_{14} respectively. On imposing condition (C_1) upon these solutions we obtain

$$K_1^{(1)}=K_2^{(1)}=K_3^{(1)}=K_5^{(1)}=K_6^{(1)}=0,$$

and from condition (C_2) we have the remaining constant

$$K_4^{(1)} = -\varepsilon\alpha_1^2 u_{12}^{(1)}(0).$$

When the constants of integration are thus determined, the solutions (38) take the form

$$\left.\begin{array}{c} p_1^{(1)} = \varepsilon\alpha_1^2(e^{-\rho_1\tau}u_{11}^{(1)} + e^{-2\rho_1\tau}u_{12}^{(1)}), \qquad q_1^{(1)} = \varepsilon\alpha_1^2(e^{-\rho_1\tau}v_{11}^{(1)} + e^{-2\rho_1\tau}v_{12}^{(1)}), \\ r_1^{(1)} = \varepsilon\alpha_1^2(e^{-\rho_1\tau}w_{11}^{(1)} + e^{-2\rho_1\tau}w_{12}^{(1)}), \end{array}\right\} \qquad (39)$$

where the functions $u_{11}^{(1)}, u_{12}^{(1)}$; $v_{11}^{(1)}, v_{12}^{(1)}$; $w_{11}^{(1)}, w_{12}^{(1)}$ are of the same form as u_{14}, v_{14}, and w_{14} respectively.

The remaining steps of the integration can be carried on in essentially the same way. By an induction to the general term we shall show that the solutions of (11) can be constructed to any desired degree of accuracy.

Let us assume that $p_1^{(\nu)}$, $q_1^{(\nu)}$, $r_1^{(\nu)}$ have been constructed for $\nu = 1, 2, \ldots, k-1$, and that

$$p_1^{(\nu)} = \varepsilon^\nu\alpha_1^{\nu+1}\sum_{l=1}^{\nu+1} e^{-l\rho_1\tau}u_{1l}^{(\nu)}, \quad q_1^{(\nu)} = \varepsilon^\nu\alpha_1^{\nu+1}\sum_{l=1}^{\nu+1} e^{-l\rho_1\tau}v_{1l}^{(\nu)}, \quad r_1^{(\nu)} = \varepsilon^\nu\alpha_1^{\nu+1}\sum_{l=1}^{\nu+1} e^{-l\rho_1\tau}w_{1l}^{(\nu)}, \qquad (40)$$

where the functions $u_{1l}^{(\nu)}$, $v_{1l}^{(\nu)}$, and $w_{1l}^{(\nu)}$ are similar to u_{14}, v_{14}, and w_{14} respectively. We propose to show that the solutions for $p_1^{(k)}$, $q_1^{(k)}$, and $r_1^{(k)}$ are the same as (40) if $\nu = k$.

Consider the set of differential equations obtained by equating the coefficients of γ^k in (11′). The left members are the same as (12A), except for the superscript k on p_1, q_1, and r_1. Let the corresponding right members be denoted by $P_1^{(k)}$, $Q_1^{(k)}$, and $R_1^{(k)}$. Then it is found that

$$P_1^{(k)} = \varepsilon^k\alpha_1^{k+1}\sum_{l=1}^{k+1} e^{-l\rho_1\tau}U_{1l}^{(k)}, \quad Q_1^{(k)} = \varepsilon^k\alpha_1^{k+1}\sum_{l=1}^{k+1} e^{-l\rho_1\tau}V_{1l}^{(k)}, \quad R_1^{(k)} = \varepsilon^k\alpha_1^{k+1}\sum_{l=1}^{k+1} e^{-l\rho_1\tau}W_{1l}^{(k)},$$

where $U_{1l}^{(k)}$, $V_{1l}^{(k)}$, and $W_{1l}^{(k)}$ are functions of the same form as u_{14}, v_{14}, and w_{14} respectively. The complementary functions of these equations are the same as at each preceding step of the integration, and the complete solutions may also be obtained in the same way as equations (38) were determined. The equations analogous to (36) and (37) differ from (36) and (37) respectively only in the superscript k instead of 1. The complete solutions will therefore have the same form as (38), except that the respective particular integrals are functions similar to $P_1^{(k)}$, $Q_1^{(k)}$, and $R_1^{(k)}$ instead of $P_1^{(1)}$, $Q_1^{(1)}$, and $R_1^{(1)}$ as in (38). From condition (C_1) the arbitrary constants will all be zero, except $K_4^{(k)}$, and from condition (C_2) this constant is found to carry the factor $\varepsilon^k\alpha_1^{k+1}$. Hence the desired solutions for $p_1^{(k)}$, $q_1^{(k)}$, and $r_1^{(k)}$ are the same as (40) if $\nu = k$. This completes the induction.

When these solutions for $p_1^{(\nu)}$, $q_1^{(\nu)}$, and $r_1^{(\nu)}$, $(\nu=1, 2, \ldots . \infty)$ are substituted in (34) and (33), we obtain as the solutions of (11),

$$\left.\begin{aligned} p_1 &= \sum_{j=1}^{\infty} \sum_{k=1}^{j} e^{-k\rho_1\tau} u_{1k}^{(j)}(\tau)\varepsilon^{j-1}\alpha_1^j\gamma^j, \\ q_1 &= \sum_{j=1}^{\infty} \sum_{k=1}^{j} e^{-k\rho_1\tau} v_{1k}^{(j)}(\tau)\varepsilon^{j-1}\alpha_1^j\gamma^j, \\ r_1 &= \sum_{j=1}^{\infty} \sum_{k=1}^{j} e^{-k\rho_1\tau} w_{1k}^{(j)}(\tau)\varepsilon^{j-1}\alpha_1^j\gamma^j, \end{aligned}\right\} \qquad (41)$$

where the superscript on the functions $u_{1k}^{(j)}(\tau)$, $v_{1k}^{(j)}(\tau)$, and $w_{1k}^{(j)}(\tau)$ has been made to conform with the corresponding power of γ. Since the two arbitrary parameters α_1 and γ occur only in products as indicated, we may suppress either without loss of generality. Let us suppose that $\alpha_1=1$.

Equations (41) are the asymptotic solutions of (11) which approach zero as τ and therefore t approach $+\infty$. In the same way there could be obtained the asymptotic solutions which approach zero as τ and therefore t approach $-\infty$. It is not necessary to construct these solutions, however, since they can be obtained directly from (41), as we proceed to show.

It was shown in the two paragraphs following equation (20) that if a set of solutions of equations (11) is known, then another set can be obtained by changing the signs of τ, q_1, and r_1 in the former set. On making these changes in (41) we thus obtain the asymptotic solutions of (11) which approach zero as τ approaches $-\infty$, viz.,

$$\left.\begin{aligned} p_1 &= +\sum_{j=1}^{\infty} \sum_{k=1}^{j} e^{k\rho_1\tau} u_{1k}^{(j)}(-\tau)\varepsilon^{j-1}\gamma^j, \\ q_1 &= -\sum_{j=1}^{\infty} \sum_{k=1}^{j} e^{k\rho_1\tau} v_{1k}^{(j)}(-\tau)\varepsilon^{j-1}\gamma^j, \\ r_1 &= -\sum_{j=1}^{\infty} \sum_{k=1}^{j} e^{k\rho_1\tau} w_{1k}^{(j)}(-\tau)\varepsilon^{j-1}\gamma^j. \end{aligned}\right\} \qquad (42)$$

Upon substituting (41) and (42) in (10), and returning to the original variables ξ, η, ζ through the substitutions (4) and (2), the asymptotic solutions of the original differential equations (1) in terms of τ are found to be

$$\left.\begin{aligned} \xi &= \xi_0+\bar{x}_1+\sum_{j=1}^{\infty} \sum_{k=1}^{j} e^{\mp k\rho_1\tau} u_{1k}^{(j)}(\pm\tau)\varepsilon^j\gamma^j, \\ \eta &= 0+\bar{y}_1\pm\sum_{j=1}^{\infty} \sum_{k=1}^{j} e^{\mp k\rho_1\tau} v_{1k}^{(j)}(\pm\tau)\varepsilon^j\gamma^j, \\ \zeta &= 0+\bar{z}_1\pm\sum_{j=1}^{\infty} \sum_{k=1}^{j} e^{\mp k\rho_1\tau} w_{1k}^{(j)}(\pm\tau)\varepsilon^j\gamma^j. \end{aligned}\right\} \qquad (43)$$

Where the double signs occur, the upper signs give the solutions which approach the periodic orbits as τ approaches $+\infty$, and the lower signs give the solutions which approach the same periodic orbits as τ approaches $-\infty$.

Asymptotic Orbits of Class B.

We consider now the orbits which are asymptotic to the periodic orbits of Class B. The method of constructing these orbits is entirely similar to the one used in constructing the asymptotic orbits of Class A.

On putting $j=2$ and considering the differential equations obtained by equating the coefficients of the various powers of γ in (11′), that is, in (11) when (33) and (34) have been substituted and γ divided out, we obtain solutions which will give the asymptotic orbits, provided the arbitrary constants of integration are so chosen that conditions (C_1) and (C_2) are satisfied.

The differential equations arising from the terms independent of γ in (11′) are the same as (12B), except for the superscript 0 on p_2, q_2, and r_2. Their solutions are therefore the same as (31) and (32). When the conditions (C_1) and (C_2) are imposed, we obtain

$$p_2^{(0)}=\alpha_2 e^{-\rho_2\tau}u_{22}, \quad q_2^{(0)}=-\alpha_2 e^{-\rho_2\tau}v_{22}, \quad r_2^{(0)}=0.$$

From the coefficients of γ to the first degree in (11′) we obtain the differential equations which define $p_2^{(1)}$, $q_2^{(1)}$, and $r_2^{(1)}$. These equations have the same left members as (12B), except the superscript 1. Let the respective right members be denoted by $P_2^{(1)}$, $Q_2^{(1)}$, and $R_2^{(1)}$. Then

$$P_2^{(1)}=\varepsilon\alpha_2^2 e^{-2\rho_2\tau}U_{22}^{(1)}, \quad Q_2^{(1)}=\varepsilon\alpha_2^2 e^{-2\rho_2\tau}V_{22}^{(1)}, \quad R_2^{(1)}=0,$$

where $U_{22}^{(1)}$ and $V_{22}^{(1)}$ are functions of the same form as u_{22} and v_{22} respectively. The complementary functions of the differential equations are the same as (31) and (32), but let the arbitrary constants be denoted by $l_1^{(1)}, \ldots, l_6^{(1)}$. By using the method of the variation of parameters to obtain the complete solutions we have

$$\left.\begin{aligned}
&\dot{l}_1^{(1)}e^{\rho_2\tau}u_{21}+\dot{l}_2^{(1)}e^{-\rho_2\tau}u_{22}+\dot{l}_3^{(1)}u_{23}+\dot{l}_4^{(1)}(u_{24}+L\tau u_{23})=0,\\
&\dot{l}_1^{(1)}e^{\rho_2\tau}(\rho_2u_{21}+\dot{u}_{21})+\dot{l}_2^{(1)}e^{-\rho_2\tau}(-\rho_2u_{22}+\dot{u}_{22})+\dot{l}_3^{(1)}\dot{u}_{23}\\
&\qquad\qquad+\dot{l}_4^{(1)}[\dot{u}_{24}+L(u_{23}+\tau\dot{u}_{23})]=P_2^{(1)},\\
&\dot{l}_1^{(1)}e^{\rho_2\tau}v_{21}-\dot{l}_2^{(1)}e^{-\rho_2\tau}v_{22}+\dot{l}_3^{(1)}v_{23}+\dot{l}_4^{(1)}(v_{24}+L\tau v_{23})=0,\\
&\dot{l}_1^{(1)}e^{\rho_2\tau}(\rho_2v_{21}+\dot{v}_{21})+\dot{l}_2^{(1)}e^{-\rho_2\tau}(\rho_2v_{22}-\dot{v}_{22})+\dot{l}_3^{(1)}\dot{v}_{23}\\
&\qquad\qquad+\dot{l}_4^{(1)}[\dot{v}_{24}+L(v_{23}+\tau\dot{v}_{23})]=Q_2^{(1)},\\
&\dot{l}_5^{(1)}e^{i\omega\tau}w_{21}+\dot{l}_6^{(1)}e^{-i\omega\tau}w_{22}=0,\\
&\dot{l}_5^{(1)}e^{i\omega\tau}(i\omega w_{21}+\dot{w}_{21})-\dot{l}_6^{(1)}e^{-i\omega\tau}(i\omega w_{22}-\dot{w}_{22})=0.
\end{aligned}\right\}\quad(44)$$

The last two equations are independent of the first four. Since the determinant of the coefficients of $\dot{l}_5^{(1)}$ and $\dot{l}_6^{(1)}$ in these last two equations is different from zero, viz., Δ_3, the only solutions for $\dot{l}_5^{(1)}$ and $\dot{l}_6^{(1)}$ are $\dot{l}_5^{(1)}=\dot{l}_6^{(1)}=0$.

The determinant of the coefficients of $\dot{l}_1^{(1)}, \ldots, \dot{l}_4^{(1)}$ is different from zero, viz., Δ_2 equation (30), and therefore the first four equations of (44) can be solved for $\dot{l}_1^{(1)}, \ldots, \dot{l}_4^{(1)}$. The resulting solutions are

$$\dot{l}_k^{(1)} = \frac{\Delta_{2k}^{(1)}}{\Delta_2}, \qquad (k=1, \ldots, 4), \tag{45}$$

where $\Delta_{2k}^{(1)}$ is the determinant obtained by replacing the elements of the k-th row of Δ_2 with 0, $P_2^{(1)}$, 0, and $Q_2^{(1)}$ respectively. Since the right members $P_2^{(1)}$ and $Q_2^{(1)}$ do not contain terms in $e^{\pm p_2\tau}$, the integrations of (45) for $l_1^{(1)}$ and $l_2^{(1)}$ will not contain terms in τ explicitly. Such terms arise, however, in the integration for $l_3^{(1)}$ and $l_4^{(1)}$, but when they are substituted in the complementary functions they cancel off. The complete solutions of the differential equations at this step are thus found to be

$$\left.\begin{aligned} p_2^{(1)} &= L_1^{(1)} e^{p_2\tau} u_{21} + L_2^{(1)} e^{-p_2\tau} u_{22} + L_3^{(1)} u_{23} + L_4^{(1)} (u_{24} + L\tau u_{23}) + \varepsilon\alpha_2^2 e^{-2p_2\tau} u_{22}^{(1)}(\tau), \\ q_2^{(1)} &= L_1^{(1)} e^{p_2\tau} v_{21} - L_2^{(1)} e^{-p_2\tau} v_{22} + L_3^{(1)} v_{23} + L_4^{(1)} (v_{24} + L\tau v_{23}) + \varepsilon\alpha_2^2 e^{-2p_2\tau} v_{22}^{(1)}(\tau), \\ r_2^{(1)} &= L_5^{(1)} e^{i\omega\tau} w_{21} + L_6^{(1)} e^{-i\omega\tau} w_{22}, \end{aligned}\right\} \tag{46}$$

where $L_1^{(1)} \ldots, L_6^{(1)}$ are the constants of integration, and $u_{22}^{(1)}(\tau)$ and $v_{22}^{(1)}(\tau)$ are functions similar to u_{21} and v_{21} respectively. From condition (C_1) we have

$$L_1^{(1)} = L_3^{(1)} = L_4^{(1)} = L_5^{(1)} = L_6^{(1)} = 0,$$

and from condition (C_2)

$$L_2^{(1)} = -\varepsilon\alpha_2^2 u_{22}^{(1)}(0).$$

Then the solutions (46) become

$$\begin{aligned} p_2^{(1)} &= \varepsilon\alpha_2^2[e^{-p_2\tau} u_{21}^{(1)}(\tau) + e^{-2p_2\tau} u_{22}^{(1)}(\tau)], \\ q_2^{(1)} &= \varepsilon\alpha_2^2[e^{-p_2\tau} v_{21}^{(1)}(\tau) + e^{-2p_2\tau} v_{22}^{(1)}(\tau)], \\ r_2^{(1)} &= 0, \end{aligned}$$

where $u_{2k}^{(1)}(\tau)$ and $v_{2k}^{(1)}(\tau)$, $(k=1, 2)$, are similar to u_{21} and v_{21} respectively.

The remaining steps of the integration can be carried on in the same way. Proceeding by induction to the general term, we find that

$$p_2^{(\nu)} = \varepsilon^\nu \alpha_2^{\nu+1} \sum_{k=1}^{\nu+1} e^{-kp_2\tau} u_{2k}^{(\nu)}(\tau), \quad q_2^{(\nu)} = \varepsilon^\nu \alpha_2^{\nu+1} \sum_{k=1}^{\nu+1} e^{-kp_2\tau} v_{2k}^{(\nu)}(\tau), \quad r_2^{(\nu)} = 0,$$

where the functions $u_{2k}^{(\nu)}(\tau)$ and $v_{2k}^{(\nu)}(\tau)$ have the same form as u_{21} and v_{21} respectively. When these terms are substituted in (34) and (33), we find the solutions of (11) to be

$$p_2 = \sum_{j=1}^{\infty} \sum_{k=1}^{j} e^{-kp_2\tau} u_{2k}^{(j)}(\tau) \varepsilon^{j-1} \alpha_2^j \gamma^j, \quad q_2 = \sum_{j=1}^{\infty} \sum_{k=1}^{j} e^{-kp_2\tau} v_{2k}^{(j)}(\tau) \varepsilon^{j-1} \alpha_2^j \gamma^j, \quad r_2 = 0. \tag{47}$$

As in the solutions (41) we may put $\alpha_2 = 1$. These equations (47) are the asymptotic solutions which approach zero as τ approaches $+\infty$. By changing the signs of τ and q_2 as in equations (42), we obtain

$$p_2=+\sum_{j=1}^{\infty}\sum_{k=1}^{j} e^{k\rho_2\tau}u_{2k}^{(j)}(-\tau)\varepsilon^{j-1}\gamma^j, \quad q_2=-\sum_{j=1}^{\infty}\sum_{k=1}^{j} e^{k\rho_2\tau}v_{2k}^{(j)}(-\tau)\varepsilon^{j-1}\gamma^j, \quad r_2=0, \tag{48}$$

which are the asymptotic solutions approaching zero as τ approaches $-\infty$. Since $r_2=0$ in (47) and (48), the asymptotic orbits are of two dimensions and are complanar with the periodic orbits of Class B.

In terms of the original variables ξ, η, ζ, these asymptotic solutions of equations (1) are

$$\left.\begin{aligned} \xi&=\xi_0+\bar{x}_2+\sum_{j=1}^{\infty}\sum_{k=1}^{j} e^{\mp k\rho_2\tau}u_{2k}^{(j)}(\pm\tau)\varepsilon^j\gamma^j, \\ \eta&=0+\bar{y}_2\pm\sum_{j=1}^{\infty}\sum_{k=1}^{j} e^{\mp k\rho_2\tau}v_{2k}^{(j)}(\pm\tau)\varepsilon^j\gamma^j, \\ \zeta&=0. \end{aligned}\right\} \tag{49}$$

These solutions approach the periodic orbits of Class B as τ approaches $+\infty$ or $-\infty$ according respectively as the upper or lower signs are taken.

§ 9. GEOMETRICAL CONSIDERATIONS.

The asymptotic solutions (43) and (49) contain the three undetermined constants t_0, ε, and γ. The constant t_0 denotes the initial time and may be put equal to zero without loss of generality. The parameters ε and γ are the respective scale factors of the periodic and asymptotic orbits. The physical interpretation of the parameter ε has already been referred to in the latter part of § 4. From the way in which the initial conditions were chosen in (C_2), it is evident that $\varepsilon\gamma$ denotes the ξ-component of the infinitesimal body's initial displacement from the periodic orbit.

Let us now consider the directions in which the asymptotic orbits approach the periodic orbits. We shall discuss only the orbits which approach the periodic orbits of Class A as τ approaches $+\infty$, since the discussion is entirely similar for the other asymptotic orbits.

As τ becomes very large, the most important terms of the solutions of (11) are those in $e^{-\rho_1\tau}$. These terms arise only from the complementary functions of the differential equations which define the various $p_1^{(k)}$, $q_1^{(k)}$, and $r_1^{(k)}$ in (34). Neglecting the explicit values of the constants of integration which are associated with $e^{-\rho_1\tau}$ at each stage of the integration, we find that the predominating terms of the solutions as τ approaches $+\infty$ are

$$\begin{aligned} p_1&=+e^{-\rho_1\tau}u_{14}[K_4^{(0)}\gamma+K_4^{(1)}\gamma^2+K_4^{(2)}\gamma^3+\ldots.], \\ q_1&=-e^{-\rho_1\tau}v_{14}[K_4^{(0)}\gamma+K_4^{(1)}\gamma^2+K_4^{(2)}\gamma^3+\ldots.], \\ r_1&=-e^{-\rho_1\tau}w_{14}[K_4^{(0)}\gamma+K_4^{(1)}\gamma^2+K_4^{(2)}\gamma^2+\ldots.], \end{aligned}$$

where $K_4^{(0)}$, $K_4^{(1)}$, ..., are the constants of integration.

The projection on the $\xi\eta$-plane of the direction of the approach is

$$\underset{\tau=+\infty}{\text{limit}} \frac{dq_1}{d\tau} \Big/ \frac{dp_1}{d\tau} = \underset{\tau=+\infty}{\text{limit}} \frac{\dot{v}_{14}-\rho_1 v_{14}}{\rho_1 u_{14}-\dot{u}_{14}}.$$

This limit is independent of γ, but, since u_{14} and v_{14} contain sines and cosines of multiples of $\sqrt{A}\tau$, it is indeterminate. The projections on the $\xi\zeta$- and $\eta\zeta$-planes are likewise found to be indeterminate. Similar results are obtained for the orbits which approach the periodic orbits of Class A as τ approaches $-\infty$, and also for the corresponding orbits of Class B.

§ 10. ILLUSTRATIVE EXAMPLES.

We shall conclude this article with numerical examples and diagrams of the periodic and asymptotic orbits which have been discussed in the preceding sections. In these examples the ratio of the finite masses is ten to one, or $1-\mu=10/11$ and $\mu=1/11$, being the ratio used in the particular periodic orbits already mentioned in § 4.

The solutions for the asymptotic orbits have been carried out to the third degree in ε for orbits of Class A and to the second degree in ε for orbits of Class B, but only to the first degree in γ for both classes of orbits.

In the numerical results that are to be found in the tables which follow, ε has the value 0.5 for Class A and 0.01 for Class B while $\gamma=0.1$ for both classes. The values of εx_j, εy_j, and εz_j $(j=1,\ 2)$ in the tables are the coordinates for the periodic orbits for the various values of τ indicated. The values for $\varepsilon\gamma p_j$, $\varepsilon\gamma q_j$, and $\varepsilon\gamma r_j$ are the amounts which must be added to the coordinates for the periodic orbits in order to obtain the asymptotic orbits. For $j=2$, $z_j=r_j=0$.

The periodic orbits are represented by the heavy lines in the diagrams, and the asymptotic orbits by the dotted lines. The arrows indicate the direction of motion. No arrows appear in the periodic orbits in Figs. 2, 5, and 8, as in these projections the infinitesimal body oscillates up and down along the same curve. The origin of coordinates is taken at the equilibrium point marked in the diagram, and the axes are parallel to the rotating $\xi\eta\zeta$ axes (see equations (2)). The unit of measurement is indicated in each diagram.

The following results have been obtained.

Orbits of Class A.

Equilibrium point (*a*).

$$\varepsilon p_1 = e^{-\rho_1\tau}\gamma[\varepsilon+\varepsilon^3(-0.29+0.29\cos 2\sqrt{A}\tau-1.08\sin 2\sqrt{A}\tau)+\ldots\ldots],$$

$$\varepsilon q_1 = e^{-\rho_1\tau}\gamma[-0.75\,\varepsilon+\varepsilon^3(0.22-0.61\cos 2\sqrt{A}\tau-0.56\sin 2\sqrt{A}\tau)+\ldots\ldots],$$

$$\varepsilon r_1 = e^{-\rho_1\tau}\gamma[\varepsilon^2(1.58\cos\sqrt{A}\tau+0.91\sin\sqrt{A}\tau)+\ldots\ldots],$$

$$\rho_1 = 1.83-0.23\,\varepsilon^2+\ldots\ldots,\quad A=2.548.$$

TABLE 3.

$\varepsilon=0.5,\ \gamma=0.1.$

τ	ϵx_1	ϵy_1	ϵz_1	$\epsilon\gamma p_1$	$\epsilon\gamma q_1$	$\epsilon\gamma r_1$
0	—.041	0	0	.0500	—.0420	.0400
.1	—.043	—.008	.043	.0385	—.0369	.0359
.2	—.047	—.015	.085	.0295	—.0309	.0315
.3	—.054	—.021	.125	.0223	—.0259	.0270
.4	—.064	—.026	.163	.0173	—.0212	.0227
.5	—.074	—.028	.198	.0136	—.0173	.0187
.6	—.084	—.028	.228	.0111	—.0142	.0155
.7	—.095	—.025	.254	.0093	—.0107	.0118
.8	—.104	—.021	.274	.0080	—.0083	.0091
.9	—.111	—.015	.289	.0071	—.0064	.0067
1.0	—.116	—.007	.297	.0065	—.0048	.0048
1.2	—.115	.009	.296	.0056	—.0029	.0020
1.4	—.103	.022	.271	.0045	—.0020	.0003
1.6	—.083	.028	.223	.0034	—.0015	—.0005 6
1.8	—.062	.028	.157	.0025	—.0013	—.0009 1
2.0	—.046	.014	.078	.0017	—.0011	—.0009 5
2.4	—.048	—.016	—.092	.0005 9	—.0006	—.0006 5
2.8	—.087	—.027	—.233	.0002 3	—.0002 7	—.0003 0
3.2	—.116	—.006	—.298	.0001 4	—.0000 9	—.0000 9
3.6	—.102	.022	—.269	.0000 92	—.0000 7	—.0000 4
4.0	—.060	.024	—.151	.0000 48	—.0000 3	—.0000 18
4.4	—.041	—.003	.014	.0000 15	—.0000 2	—.0000 16

The projections of the above orbits on the coordinate planes are given in Figs. 1, 2, and 3.

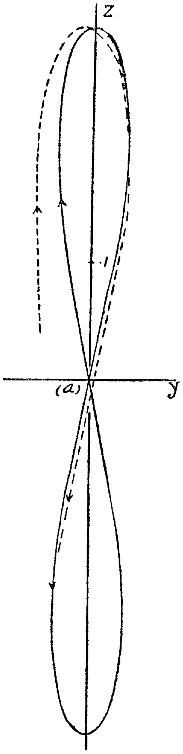

FIG. 1.

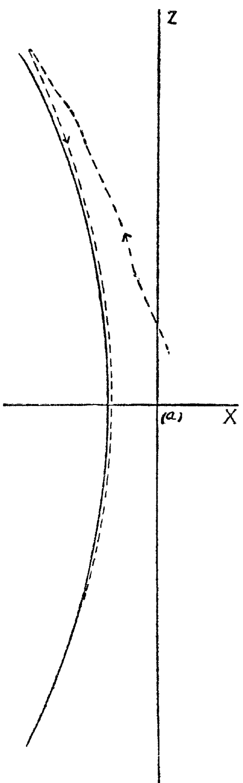

FIG. 2.

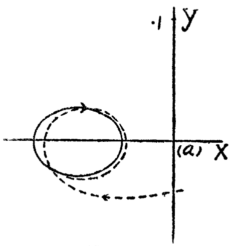

FIG. 3.

Equilibrium point (b).

$$\varepsilon p_1 = e^{-\rho_1\tau}\gamma[\varepsilon + \varepsilon^3(0.222 - 0.222\cos 2\sqrt{A}\tau - 0.454\sin 2\sqrt{A}\tau) + \ldots.],$$
$$\varepsilon q_1 = e^{-\rho_1\tau}\gamma[0.40\,\varepsilon + \varepsilon^2(0.089 - 0.182\cos 2\sqrt{A}\tau - 0.037\sin 2\sqrt{A}\tau) + \ldots.],$$
$$\varepsilon r_1 = e^{-\rho_1\tau}\gamma[\varepsilon(-0.345\sin\sqrt{A}\tau - 0.523\cos\sqrt{A}\tau) + \ldots.],$$
$$\rho_1 = 3.366 - 2.837\,\varepsilon^2 + \ldots.,\quad A = 6.510.$$

TABLE 4.

$\varepsilon = 0.5,\quad \gamma = 0.1.$

τ	ϵx_1	ϵy_1	ϵz_1	$\epsilon\gamma p_1$	$\epsilon\gamma q_1$	$\epsilon\gamma r_1$
0	.0135	0	0	.0500	.0188	—.0131
.1	.0147	.0022	.0474	.0300	.0119	—.0094 0
.2	.0178	.0038	.0917	.0186	.0073 8	—.0062 6
.3	.0221	.0045	.1355	.0118	.0051 7	—.0038 6
.4	.0266	.0040	.1602	.0077 9	.0034 5	—.0022 6
.5	.0300	.0025	.1799	.0051 9	.0022 7	—.0009 57
.6	.0315	.0004	.1878	.0034 9	.0014 7	—.0005 76
.7	.0307	—.0019	.1837	.0023 1	.0009 36	—.0002 24
.8	.0278	—.0036	.1677	.0014 9	.0005 74	—.0000 454
.9	.0236	—.0045	.1594	.0009 34	.0003 47	+.0000 350
1.0	.0206	—.0044	.1047	.0005 83	.0002 13	+.0000 616
1.2	.0136	—.0007	.0150	.0002 04	.0000 762	+.0000 497
1.4	.0166	.0034	—.0784	.0000 750	.0000 309	+.0000 363
1.6	.0252	.0043	—.1519	.0000 308	.0000 136	+.0000 0936
1.8	.0312	.0011	—.1867	.0000 137	.0000 0590	+.0000 0257
2.0	.0289	—.0032	—.1837	.0000 0594	.0000 0232	+.0000 0056 6
2.2	.0205	—.0044	—.1169	.0000 0233	.0000 0081 0	—.0000 0021 6
2.4	.0140	—.0014	—.0301	.0000 0086 6	.0000 0030 8	—.0000 0002 2
2.6	.0156	.0029	.0645	.0000 0029 9	.0000 0012 4	—.0000 0001 9

The projections of the above orbits on the coordinate planes are given in Figs. 4, 5, and 6.

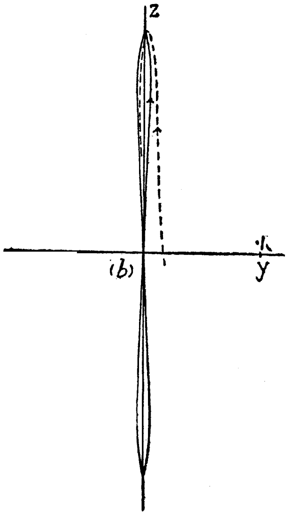

FIG. 4.

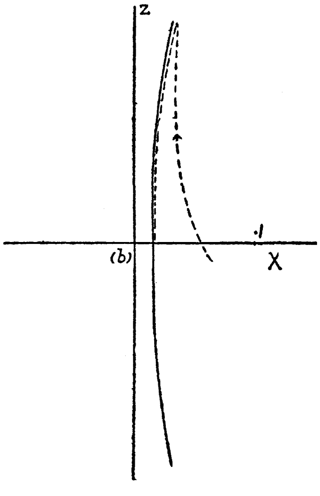

FIG. 5.

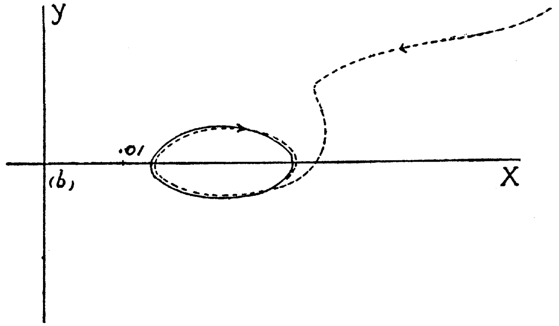

FIG. 6

Equilibrium point (c).

$$\varepsilon p_1 = e^{-\rho_1\tau}\gamma[\varepsilon + \varepsilon^3(-0.104 + 0.104\cos 2\sqrt{A}\tau - 0.817\sin 2\sqrt{A}\tau) + \ldots.],$$
$$\varepsilon q_1 = e^{-\rho_1\tau}\gamma[3.08\,\varepsilon + \varepsilon^3(-0.320 - 0.787\cos 2\sqrt{A}\tau - 0.106\sin 2\sqrt{A}\tau) + \ldots.],$$
$$\varepsilon r_1 = e^{-\rho_1\tau}\gamma[\varepsilon(-0.719\sin\sqrt{A}\tau - 3.145\cos\sqrt{A}\tau) + \ldots.],$$
$$\rho_1 = 0.457 - 0.025\,\varepsilon^2 + \ldots.,\quad A = 1.082.$$

TABLE 5.

$\varepsilon=0.5, \quad \gamma=0.1.$

τ	ϵx_1	ϵy_1	ϵz_1	$\epsilon\gamma p_1$	$\epsilon\gamma q_1$	$\epsilon\gamma r_1$
0	.0050	0	0	.0500	.1402	—.0786
.2	.0099	.0228	.0989	.0418	.1279	—.0733
.4	.0238	.0418	.1939	.0349	.1180	—.0657
.6	.0443	.0536	.2803	.0297	.1098	—.0560
.8	.0678	.0563	.3547	.0264	.1030	—.0455
1.0	.0905	.0493	.4138	.0245	.0960	—.0346
1.2	.1084	.0340	.4550	.0236	.0893	—.0239
1.4	.1185	.0129	.4766	.0234	.0825	—.0139
1.6	.1190	—.0105	.4781	.0233	.0754	—.0051
1.8	.1099	—.0320	.4584	.0229	.0680	.0026
2.0	.0927	—.0481	.4190	.0221	.0612	.0088
2.2	.0703	—.0560	.3619	.0209	.0543	.0135
2.4	.0466	—.0543	.2890	.0191	.0481	.0168
2.6	.0256	—.0434	.2035	.0169	.0426	.0187
2.8	.0110	—.0251	.1094	.0147	.0377	.0194
3.0	.0051	—.0025	.0105	.0123	.0341	.0191
3.4	.0220	.0400	—.1843	.0086	.0289	.0161
3.8	.0653	.0564	—.3475	.0064 5	.0251	.0113
4.2	.1069	.0359	—.4517	.0057 3	.0218	.0060
4.6	.1194	—.0080	—.4790	.0056 2	.0184	.0014
5	.1101	—.0318	—.4723	.0051 0	.0151	.0015
5.4	.0491	—.0549	—.2971	.0046 6	.0119	—.0040
5.8	.0121	—.0272	—.1195	.0035 9	.0093	—.0047
6.2	.0081	.0182	.0787	.0014 5	.0076	—.0044

The projections of the above orbits on the coordinate planes are given in Figs. 7, 8, and 9.

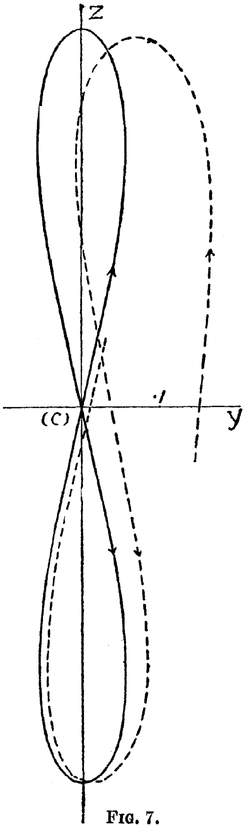

FIG. 7.

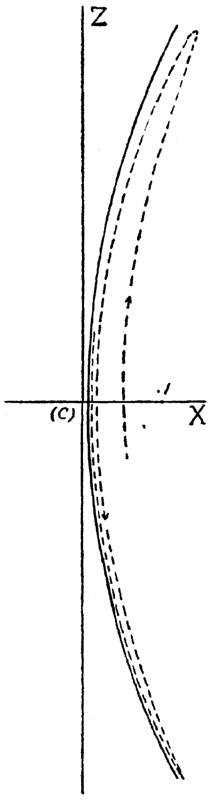

FIG. 8.

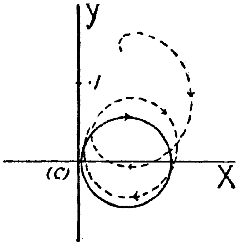

FIG. 9.

Equilibrium point (*a*).

$$\varepsilon p_2 = e^{-\rho_2\tau}\gamma[\varepsilon+\varepsilon^2(-1.20+1.20\cos\sigma\tau+3.95\sin\sigma\tau)+\ldots.],$$
$$\varepsilon q_2 = e^{-\rho_2\tau}\gamma[0.75\,\varepsilon-\varepsilon^2(0.90+4.65\cos\sigma\tau+2.46\sin\sigma\tau)+\ldots.],$$
$$\rho_2 = 1.833+(\;)\varepsilon^2+\ldots.,\quad \sigma=1.68.$$

TABLE 6.

$\varepsilon=0.01,\quad \gamma=0.1.$

τ	ϵx_2	ϵy_2	$\epsilon\gamma p_2$	$\epsilon\gamma q_2$
0	.0100	0	.0010 00	.0007 13
.2	.0094	—.0088	.0007 00	.0004 77
.4	.0078	—.0166	.0004 90	.0003 31
.6	.0053	—.0225	.0003 42	.0002 31
.8	.0023	—.0259	.0002 38	.0001 63
1.0	—.0011	—.0264	.0001 63	.0001 16
1.2	—.0043	—.0240	.0001 13	.0000 820
1.4	—.0070	—.0189	.0000 776	.0000 581
1.6	—.0090	—.0117	.0000 534	.0000 412
1.8	—.0099	—.0031	.0000 361	.0000 289
2.0	—.0098	.0058	.0000 248	.0000 203
2.2	—.0085	.0140	.0000 169	.0000 140
2.4	—.0063	.0207	.0000 117	.0000 0970
2.6	—.0034	.0250	.0000 0809	.0000 0664
2.8	—.0001	.0266	.0000 0560	.0000 0452
3.0	.0032	.0252	.0000 0392	.0000 0306
3.2	.0062	.0210	.0000 0275	.0000 0208
3.4	.0084	.0144	.0000 0193	.0000 0141
3.6	.0097	.0062	.0000 0136	.0000 0095
3.8	.0100	.0027	.0000 0094	.0000 0065

These orbits are shown in Fig. 10.

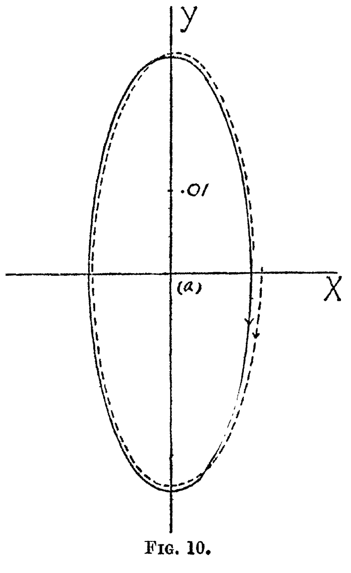

FIG. 10.

Equilibrium point (b).

$$\epsilon p_2 = e^{-\rho_2\tau}\gamma[\epsilon+\epsilon^2(0.69-0.69\cos\sigma\tau-2.47\sin\sigma\tau)+\ldots.],$$
$$\epsilon q_2 = e^{-\rho_2\tau}\gamma[0.40\,\epsilon+\epsilon^2(0.28+4.60\cos\sigma\tau+3.07\sin\sigma\tau)+\ldots.],$$
$$\rho_2=3.366+(\ )\epsilon^2+\ldots.,\qquad \sigma=2.61.$$

TABLE 7.

$\epsilon=0.01$, $\gamma=0.1$.

τ	ϵx_2	ϵy_2	$\epsilon\gamma p_2$	$\epsilon\gamma q_2$
0	.0100	0	.0010 00	.0004 49
.1	.0097	—.0103	.0007 10	.0003 70
.2	.0087	—.0199	.0005 04	.0002 34
.3	.0071	—.0282	.0003 59	.0001 66
.4	.0050	—.0345	.0002 55	.0001 17
.5	.0026	—.0385	.0001 82	.0000 83
.6	.0000 4	—.0399	.0001 31	.0000 58
.7	—.0025	—.0386	.0000 933	.0000 398
.8	—.0050	—.0347	.0000 670	.0000 275
.9	—.0070	—.0284	.0000 481	.0000 190
1.0	—.0086	—.0202	.0000 345	.0000 131
1.2	—.0100	—.0003	.0000 178	.0000 063
1.4	—.0087	.0196	.0000 092	.0000 031
1.6	—.0051	.0344	.0000 047	.0000 016
1.8	—.0001	.0399	.0000 024	.0000 0087
2.0	+.0049	.0348	.0000 012	.0000 0047
2.2	+.0086	.0204	.0000 006	.0000 0026
2.4	.0100	.0006	.0000 003	.0000 0014

These orbits are shown in Fig. 11.

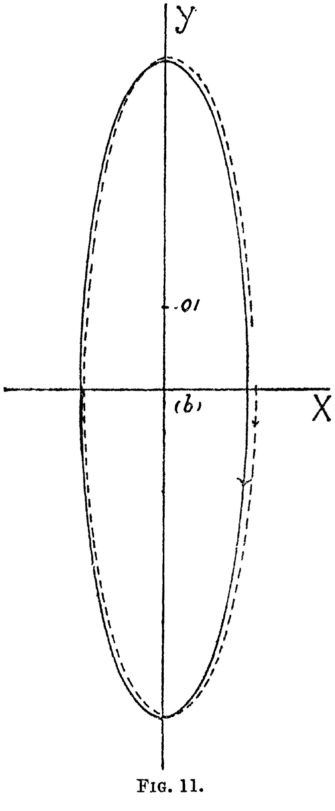

FIG. 11.

Equilibrium point (*c*).

$$\varepsilon p_2 = e^{-\rho_2\tau}\gamma[\varepsilon+\varepsilon^2(25-25\cos\sigma\tau-23\sin\sigma\tau)+\ldots.],$$
$$\varepsilon q_2 = e^{-\rho_2\tau}\gamma[3.08\,\varepsilon+\varepsilon^2(77-56\cos\sigma\tau+31\sin\sigma\tau)+\ldots.],$$
$$\rho_2=0.475+(\,)\varepsilon^2+\ldots., \quad \sigma=1.07.$$

TABLE 8.

$\varepsilon=0.01, \quad \gamma=0.1.$

τ	εx_2	εy_2	$\varepsilon\gamma p_2$	$\varepsilon\gamma q_2$
0	.0100	0	.0010 0	.0032 9
.1	.0099	—.0022	.0009 3	.0031 7
.2	.0098	—.0043	.0008 7	.0030 6
.3	.0095	—.0063	.0008 1	.0029 7
.4	.0091	—.0083	.0007 6	.0028 7
.5	.0086	—.0103	.0007 2	.0027 9
.6	.0080	—.0120	.0006 8	.0027 0
.7	.0073	—.0137	.0006 5	.0026 2
.8	.0066	—.0152	.0006 3	.0025 4
.9	.0057	—.0165	.0006 0	.0024 6
1.0	.0048	—.0176	.0005 8	.0023 9
1.5	—.0003	—.0199	.0005 4	.0020 5
2.0	—.0054	—.0169	.0004 6	.0017 1
2.5	—.0089	—.0090	.0004 2	.0013 7
3.0	—.0100	.0014	.0003 7	.0010 6
3.5	—.0082	.0140	.0003 0	.0007 8
4.0	—.0042	.0183	.0002 3	.0005 7
4.5	.0010	.0200	.0001 4	.0004 1
5.0	.0060	.0162	.0001 2	.0003 0
5.5	.0092	.0078	.0000 8	.0002 3
6.0	.0099	—.0027	.0000 6	.0001 9

These orbits are shown in Fig. 12.

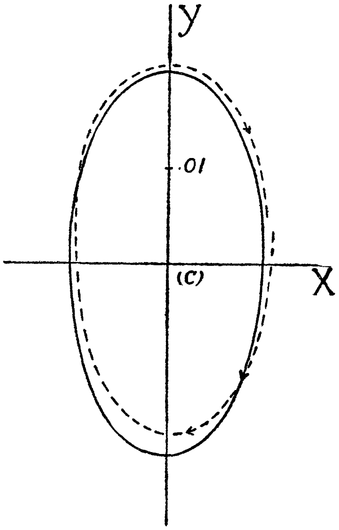

FIG. 12.

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